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Harmonic Spinors

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INTRODUCTION

With the introduction of general relativity, it became necessary to express the differential operators of mathematical physics in a coordinate-free form. This made it possible to define those operators on an arbitrary Riemannian manifold—the grads, divs, and curls got translated into the $d + d^*$ operator on the bundle of exterior forms. This particular operator found fruitful application in the theorem of Hodge which expressed the dimension of the null space (the space of harmonic forms) on a compact manifold in terms of topological invariants—the Betti numbers.

Another operator—the Dirac operator—made a later appearance in Riemannian geometry. It was used by Atiyah and Singer to explain the integrality of the $\hat{A}$-genus of a spin manifold, and then Lichnerowicz proved a strong vanishing theorem— if a spin manifold has positive scalar curvature, the null space of the Dirac operator (the space of harmonic spinors) is zero. Bearing in mind the formal similarity between the Dirac operator and the $d + d^*$ operator, one may ask if there is an analogue of Hodge's theorem—can we express the dimension of the null space in terms of topological invariants of the manifold? The main purpose of this paper is to show that this is impossible and in general the dimension of the space of harmonic spinors depends on the metric used to define the Dirac operator.

Sections 1.1–1.4, deal with what can be said in general differential geometric terms about harmonic spinors—which is very little. We show that the Dirac operator is conformally invariant in a certain sense (a fact known to physicists) and thus the dimension h of the space of harmonic spinors is invariant under a conformal change of metric. We also consider what happens to harmonic spinors if the scalar curvature is identically zero. However, to get more information, we need to consider specific examples of harmonic spinors. The first examples of naturally occurring Riemannian manifolds which spring to mind are homogeneous spaces, but a little thought shows that (except for the torus) with their natural metric they have positive scalar curvature and by Lichnerowicz, no harmonic spinors. In Sections 2.1–2.4 we look at another source of manifolds—algebraic geometry.

On a complex manifold, the spin structures are in one-to-one correspondence with holomorphic square roots of the canonical bundle K, that is, holomorphic line bundles L such that $L \otimes L \cong K$. For a Kähler manifold we can then identify the space of harmonic spinors with the holomorphic cohomology $H^*(X, \mathcal{O}(L))$. We now start looking for examples among algebraic curves and it turns out that for genus $\geqslant 3$, the dimension of the space of harmonic spinors varies with the conformal structure. Hyperelliptic curves are distinguished by special properties of their harmonic spinors. We also consider simply connected algebraic surfaces and compute several examples, but unfortunately find no examples of variation of h. The case of algebraic curves is unsatisfactory since, apart from the complication of having several spin structures, we have the additional property that h is bounded by the topological invariant $(g + 1)$. For algebraic surfaces, we also have an upper bound $b_1 + (5\tau + 4\chi)/8$, and, in general, we should expect boundedness for an algebraic family of complex structures. In Sections 3.1–3.3 we have an example of a family of Riemannian structures where boundedness no longer holds.

We consider the three-dimensional sphere S^3. Relative to the $S^3 \times S^3$-invariant metric, this of course has positive scalar curvature and no harmonic spinors. The $S^3 \times S^1$-invariant metrics are parametrized up to a constant multiple by a positive real number λ. For a generic λ, there are still no harmonic spinors but for certain values they do exist. To find the precise dimension is a number theoretical problem, but we can find enough to show that as λ varies the dimension is unbounded.

In Sections 4.1–4.5 we consider higher dimensions. The strongest result we have is the following: we can change the dimension of the space of harmonic spinors (for some spin structure) on any $8k - 1$ dimensional spin manifold by altering the metric in a neighborhood of a point. Despite the deceptive local content of this statement, we prove it by using global differential topology. We use the Atiyah–Singer index theorem for families of operators. Let $X^m \to Z \to Y$ be a differentiable fibre bundle of spin manifolds. We introduce a family of metrics in the fibres and then the Dirac operator in the fibres has an index in $KR^{-m}(Y)$. If $m = 8k - 1$, we can regard the Dirac operator as a real self-adjoint operator and then if the family of null spaces has constant rank, this index is zero, which implies the vanishing of a certain KR-characteristic number of Z. Now by using an exotic sphere which is not a spin boundary and the result of Cerf on pseudo-isotopy, we construct examples $X \to Z \to S^2$ for which this number is nonzero and deduce the above result. The exotic spheres used are of interest to differential geometers as they do not admit metrics of positive scalar curvature. However, we also use them to give information on the nontriviality of the topology of the space $\mathscr{R}^+(X)$ of metrics of positive scalar curvature on X. In particular, we show that if X is a spin manifold of dimension $8k$ and $\mathscr{R}^+(X) \neq \varnothing$, then $\pi_i(\mathscr{R}^+) \neq 0$ for $i = 0$ and 1. Using this setup in the reverse direction, we conclude with an index-theory proof of the invariance of the Todd genus under blowing-up.

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1.1. Preliminary Definitions

For details on Clifford algebras and the spin representation, we refer to Atiyah, Bott, and Shapiro [10] and Jacobson [22].

Let U be a finite dimensional vector space over $\mathbb{R}$ and (x, y) a positive definite quadratic form on U. Then factoring out the ideal generated by elements of the form $x \otimes x + (x, x)1$ in the tensor algebra $\otimes U$, we get a finite dimensional algebra $C(U)$, the *Clifford algebra* of U. We have $U \subset C(U)$ such that $x^2 = -(x, x)1$. Suppose $\dim U = 2m$, then the complexification $C(U) \otimes_{\mathbb{R}} \mathbb{C}$ is a matrix algebra, i.e., End S, where S is a 2^m-dimensional complex vector space.

The special orthogonal group $SO(U)$ acts on U preserving the quadratic form and so induces an automorphism of $C(U) \otimes_{\mathbb{R}} \mathbb{C}$, which being a matrix algebra is an inner automorphism. We thus have

$$g \cdot \alpha = \rho(g)\, \alpha \rho(g)^{-1} \qquad (g \in SO(U);\ \alpha, \rho(g) \in \operatorname{End} S),$$

and $g \mapsto \rho(g)$ defines a two-valued representation of $SO(U)$ which lifts to a single-valued representation of the double covering *Spin* (U). The representation is not irreducible: if $\{e_1, ..., e_{2m}\}$ is an orthonormal basis for U, then $\omega = e_1 \cdots e_{2m} \in C(U)$ satisfies $\omega^2 = (-1)^m$ and commutes with the action of Spin(U). The eigenspaces S^+, S^- of ω are irreducible representation spaces and since $x\omega = -\omega x$ ($x \in U$), multiplication by x gives an isomorphism $x: S^+ \to S^-$ as vector spaces.

Spin($2m-1$) ($=$ Spin($\mathbb{R}^{2m-1}$)) is a subgroup of Spin($2m$) and acting on S commutes with multiplication by e_{2m}. Hence, $e_{2m}: S^+ \to S^-$ defines an isomorphism of representation spaces of Spin($2m-1$). S^+ is then an irreducible representation space of Spin($2m-1$).

Let $x_1, \ldots, x_p \in U$. Then define $[x_1, \ldots, x_p] \in C(U)$ inductively by the following formulas:

$$[x_1] = x_1,$$

$$[x_1, \ldots, x_{2k-1}, x_{2k}] = [[x_1, \ldots, x_{2k-1}]\, x_{2k}],$$

$$[x_1, \ldots, x_{2k}, x_{2k+1}] = [x_1, \ldots, x_{2k}] \cdot x_{2k+1},$$

where $[ab] = ab - ba$ and $a \cdot b = \frac{1}{2}(ab + ba)$.

Then for any permutation σ of $(1, 2, \ldots, p)$ $[x_{\sigma(1)}, \ldots, x_{\sigma(p)}] = \operatorname{sgn} \sigma [x_1, \ldots, x_p]$, and so if U_p denotes the subspace of $C(U)$ spanned by all the elements $[x_1, \ldots, x_p]$, we have a natural isomorphism of vector spaces $U_p \cong \lambda^p U$, the pth exterior product.

We have $[y[xz]] = 4((x, y)z - (y, z)x)$, and so the restriction of the adjoint representation of $C(U)$ (as a Lie algebra) to U_2 leaves $U_1 (= U)$ stable and acts as an element of the orthogonal Lie algebra $L(SO(U))$. Hence on the Lie algebra level, the spin representation is given by: $L(SO(U)) \ni z \otimes x - x \otimes z \mapsto \frac{1}{4}[x, z] \in C(U) \subset L(GL(S))$.

Let X be a compact, oriented *riemannian* manifold, i.e., we have a positive definite quadratic form on the tangent bundle T. The bundle of orthonormal frames E is a principal SO-bundle. Suppose E lifts to give a principal Spin-bundle $\tilde{E}$; then X is a *spin manifold* and we can define via the spin representation a vector bundle $V = \tilde{E} \times_{\text{Spin}} S$, the *bundle of spinors*.

E lifts to $\tilde{E}$ iff $w_2(X) = 0$ and any two liftings differ by a $\mathbb{Z}_2$ 1-cocycle, so the number of inequivalent liftings (the number of *spin structures*) is $\# H^1(X, \mathbb{Z}_2)$.

The riemannian connection induces a connection on V—if the connection matrix in T is locally given by ω_{ij} relative to an orthonormal basis $\{e_1, \ldots, e_n\}$, then relative to the corresponding spinor basis defined by the lifting, the lifted connection matrix is, from the previous discussion, given by $\frac{1}{4} \sum_{i,j} \omega_{ij} e_i e_j \in \Gamma(\operatorname{End} V \otimes T^*)$ locally.

From the vector space isomorphism between the Clifford algebra and the exterior algebra, we can regard (via the duality $T^* \cong T$ defined by the metric) an exterior form on X as an endomorphism of the spinor bundle V by Clifford multiplication.

We define the *Dirac operator* P by the composition

$$\Gamma(V) \xrightarrow{D} \Gamma(V \otimes T^*) \xrightarrow{m} \Gamma(V),$$

where D is the covariant derivative relative to the induced connection on V and m is Clifford multiplication by an element of T^*. (Locally $P\psi = \sum e_i \nabla_i \psi$, where $\nabla_i \psi$ is covariant differentiation in the direction e_i.) The operator P^2 is the *spinor laplacian* and since P is self-adjoint, the two operators have the same null space.

P is an elliptic differential operator and $\ker P = H$ is the finite dimensional space of *harmonic spinors*. Corresponding to the irreducible representation spaces S^+, S^-, we have a decomposition $V = V^+ \oplus V^-$ and the Dirac operator takes sections of V^+ into sections of V^-. We then get a decomposition $H = H^+ \oplus H^-$ where H^+ is the space of *positive* harmonic spinors and H^- the space of negative ones. If dim X is odd, we usually consider the Dirac operator $e_{2m}P : \Gamma(V^+) \to \Gamma(V^+)$.

Let $\mathrm{Spin}^c(U) = \mathrm{Spin}(U) \times_{\mathbb{Z}_2} S^1$, then we have the following exact sequences:

$$1 \longrightarrow S^1 \longrightarrow \mathrm{Spin}^c \to SO \to 1$$

$$1 \to \mathrm{Spin} \to \mathrm{Spin}^c \to S^1 \to 1.$$

A manifold X is a Spin^c manifold if E lifts to a principal Spin^c bundle via the first sequence. From the second sequence, a Spin^c structure defines a principal S^1 bundle—equivalently a complex hermitian line bundle L. The spin representation extends to Spin^c and we can construct a Dirac operator in this situation too. The main differences are

(i) X is a Spin^c manifold iff $W_3(X) = 0$ and two Spin^c structures differ by an element of $H^2(X, \mathbb{Z})$.

(ii) To put a connection on the Spin^c bundle, we have to *choose* a hermitian connection on the line bundle L.

1.2. The Vanishing Theorem

Let X be a Spin^c manifold, and let $i\theta$ be the curvature form of the associated line bundle L. Then $\theta \in \Gamma(\lambda^2 T^*)$ defines via the riemannian metric a skew-symmetric endomorphism of T. Suppose its eigenvalues are $\pm i\lambda_1, \ldots, \pm i\lambda_m$ (where $\dim X = 2m$ or $2m + 1$); then we have the following version of the vanishing theorem of Lichnerowicz [25]:

THEOREM 1.1. *Let X be a* Spin^c *manifold with scalar curvature $R \geqslant 4\sum |\lambda_i|$ and strict inequality at some point. Then X admits no harmonic spinors.*

Proof. The Dirac operator P is given by the composition $\Gamma(V) \to^D \Gamma(V \otimes T^*) \to^m \Gamma(V)$, where m is Clifford multiplication. Now D commutes with m and hence $P^2\psi = m^2D^2\psi$, where $m^2 : \Gamma(V \otimes T^* \otimes T^*) \to \Gamma(V)$ is defined by $m^2(\psi \otimes a \otimes \beta) = a \cdot \beta \cdot \psi$.

But under the identification of the Clifford algebra with the exterior algebra,

$$C(T^*) \ni \alpha \cdot \beta = \alpha \wedge \beta - (\alpha, \beta) \in (\lambda^2 \oplus \lambda^0)(T^*).$$

Hence, since the riemannian connection has no torsion,

$$P^2\psi = \Omega \cdot \psi - \operatorname{tr} D^2\psi,$$

where $\Omega \in \Gamma(\operatorname{End} V \otimes \lambda^2 T^*)$ is the curvature form of the Spin^c connection and acts via

$$\operatorname{End} V \otimes \lambda^2 T^* \to \operatorname{End} V \otimes \operatorname{End} V \to \operatorname{End} V$$

and $\operatorname{tr} : V \otimes T^* \otimes T^* \to V$ denotes contraction via the riemannian metric:

$$V \otimes T^* \otimes T^* \to V \otimes T^* \otimes T \to V.$$

Now since $D(D\psi, \psi) = (D^2\psi, \psi) + (D\psi, D\psi) \in \Gamma(T^* \otimes T^*)$, we have

$$(\operatorname{tr} D^2\psi, \psi) = -\langle D\psi, D\psi\rangle + d^*(D\psi, \psi),$$

where $\langle\ ,\ \rangle$ denotes the inner product on $V \otimes T^*$ and $d^*: \Gamma(T^*) \to \Gamma(1)$ is the usual adjoint of the exterior derivative d.

So $(P^2\psi, \psi) = (\Omega \cdot \psi, \psi) + \langle D\psi, D\psi\rangle - d^*(D\psi, \psi)$, and integrating over X we get

$$\int_X (P^2\psi, \psi) * 1 = \int_X (\Omega \cdot \psi, \psi) + \langle D\psi, D\psi\rangle * 1.$$

$\langle D\psi, D\psi\rangle \geqslant 0$ so if $(\Omega \cdot \psi, \psi) \geqslant 0$ and $P^2\psi = 0$, then $D\psi = 0$. Thus if $\Omega > 0$ at some point, $\psi = 0$ at that point and since $D\psi = 0$, $\psi = 0$

everywhere. In order to prove the theorem, it remains to determine the endomorphism Ω.

$\Omega \in \Gamma(L(\tilde{E}) \otimes \lambda^2 T^*)$ where $L(\tilde{E})$ is the vector bundle associated to the principal Spin^c bundle $\tilde{E}$ by the adjoint representation.

$L(\text{Spin}^c) \simeq L(SO) \oplus \mathbb{R}$ where $\mathbb{R}$ is acted on trivially by the adjoint representation, so $\Omega = \Omega_0 + \Omega_1$ where $\Omega_1 \in \Gamma(L(E) \otimes \lambda^2 T^*)$ and $\Omega_0 \in \Gamma(\lambda^2 T^*)$. Ω_1 is the riemannian curvature form, $2i\Omega_0$ is the curvature form of L.

(i) Under the Spin^c representation, $S^1 \subset \text{Spin}^c$ acts as unit scalars, hence Ω_0 acts as $i \times$ Clifford multiplication. Now relative to some local orthonormal basis $\{e_1, ..., e_n\}$, Ω_0 may be written as $\sum \lambda_k e_{2k-1} \wedge e_{2k}$. We shall show that the eigenvalues of this considered as an endomorphism of V are $\sum \pm \lambda_k$.

Let $E_k = e_{2k-1} \cdot e_{2k}$ in the Clifford algebra. Then $E_k{}^2 = -1$, so E_k has eigenvalues $\pm i$. Since $E_k e_{2k} = -e_{2k-1} = -e_{2k} E_k$, multiplication by e_{2k} interchanges the eigenspaces which therefore have the same dimension. The Clifford algebra generated by $e_3, ..., e_{2m}$ commutes with E_1 and hence acts as the endomorphisms of each eigenspace of E_1. By induction, we see that V has a local basis of 2^m spinors $\psi(k_1, ..., k_m)$ (where $k_j = \pm 1$) such that E_j acts as ik_j on $\psi(k_1, ..., k_m)$. Then the eigenvalues of $\Omega_0 = i \sum \lambda_k E_k$ are $(\pm\lambda_1 \pm \lambda_2 \cdots \pm \lambda_m)$.

In particular, the smallest eigenvalue is $-\sum |\lambda_i|$.

(ii) $\Omega_1 \in \Gamma(L(E) \otimes \lambda^2 T^*) \simeq \Gamma(\lambda^2 T^* \otimes \lambda^2 T^*)$. Relative to a local orthonormal basis $\{e_1, ..., e_n\}$, $(\Omega_1)_{ij} = \frac{1}{2} \sum R_{ijk\ell} e_k \wedge e_\ell$, so the action of Ω_1 is given by $\frac{1}{2} \cdot \frac{1}{4} \cdot \sum R_{ijk\ell} e_i e_j e_k e_\ell$ i.e., the Clifford multiplication

$$\lambda^2 T^* \otimes \lambda^2 T^* \to (\lambda^0 \oplus \lambda^2 \oplus \lambda^4)(T^*) \to \text{End } V.$$

Now from the Bianchi identity,

$$R(X, Y)Z + R(Z, X)Y + R(Y, Z)X = 0,$$

the λ^4 component is zero. From the symmetry $(R(X, Y)Z, T) = (R(Z, T)X, Y)$, the λ^2 component is zero. The λ^0 component is easily seen to be $\frac{1}{4}R$.

Hence the endomorphism Ω is positive iff $\frac{1}{4}R \geqslant \sum |\lambda_i|$, which proves the theorem.

If X is a spin manifold, we can take $L = 0$, and then we retrieve the vanishing theorem of Lichnerowicz: If the scalar curvature is $\geqslant 0$ and not identically zero, there are no harmonic spinors.

Examples. (1) Let $X = G/H$, a compact homogeneous space. If $L(G) = M \oplus L(H)$ is the corresponding decomposition of the Lie algebra and we take the metric on X induced by the *bi*-invariant metric B on G, the scalar curvature is given by:

$$R_0 = \tfrac{1}{4} \sum_{i,j} B([X_i, X_j]_M, [X_i, X_j]_M) + B([X_i, X_j]_{L(H)}, [X_i, X_j]_{L(H)}),$$

where $\{X_i\}$ is an orthonormal basis for M (see Kobayashi and Nomizu [23, p. 203]). Hence $R_0 > 0$ unless X is a torus. Thus there are no harmonic spinors on a compact homogeneous space which is a spin manifold, relative to the *bi*-invariant metric.

(2) $X = \mathbb{C}P^n$. X is a Spin^c manifold. $H^2(X, \mathbb{Z})$ is generated by one element H, so suppose the line bundle L associated to the Spin^c structure is given by $L = kH$. Using the almost complex structure, the Ricci tensor S defines a 2-form ρ and $\frac{1}{2}i\rho$ is the curvature form which represents the first Chern class $c_1(X) = (n + 1)H$. We can therefore take a connection on L such that the curvature form is $ki\rho/2(n + 1)$.

X is an Einstein manifold, and the eigenvalues of ρ are $\pm i\lambda$ with multiplicity n. Furthermore, the scalar curvature $R = \operatorname{tr} S = 2n\lambda$ ($\lambda > 0$). Hence,

$$\begin{aligned} R - 4 \sum | \lambda_i | &= 2n\lambda - 4n | k | \lambda/2(n + 1) \\ &= 2n\lambda((n + 1) - | k |)/(n + 1). \end{aligned}$$

Thus, if $| k | < n + 1$, there are no harmonic spinors with respect to the standard metric.

1.3. Parallel Spinors

If X is a spin manifold and the scalar curvature $R = 0$, then the vanishing theorem says that $P\psi = 0$ implies that $D\psi = 0$; in other words, every harmonic spinor is *parallel*. The following theorem shows that parallel spinors are not very common.

THEOREM 1.2. *Let X be a compact simply connected spin manifold which admits a parallel spinor. Then if* dim X *is even* (*resp. odd*), $\pm X$ (*resp.* $\pm X \times S^1$) *is a Kähler manifold with vanishing Ricci tensor.* (*There are no known examples—see Kobayashi and Nomizu* [23, pp. 151, 175]).

Proof. If X admits a parallel spinor ψ, then the linear holonomy group $\Phi \subset SO$ leaves fixed a vector under the spin representation Δ, i.e., we can reduce the holonomy group to the identity component of an isotropy subgroup $G \subset \mathrm{Spin}$ of the spin representation. Any parallel spinor is the sum of a positive and negative parallel spinor, so we consider the irreducible representations $\Delta^{\pm}$. Also, since a change of orientation interchanges the positive and negative spinor bundles, we need only consider Δ^+.

LEMMA. *Let* $G(\Delta^+) \subset \mathrm{Spin}(2m)$ *be an isotropy subgroup of* Δ^+. *Then* $G_0(\Delta^+) \cong SU(m)$.

Proof. Suppose $g \in \mathrm{Spin}(2m)$ leaves fixed a vector ψ, i.e., $g\psi = \psi$. Then $hgh^{-1}h\psi = h\psi$, so by conjugation we can consider ψ as an eigenvector of the standard maximal torus and the subtorus which leaves ψ fixed is given by the vanishing of a weight w. The weights of Δ^+ are $\frac{1}{2}(\pm x_1 \pm x_2 + \cdots \pm x_m)$ with an even number of minus signs, where $x_1, \ldots, x_m$ are the basic characters of $SO(2m)$. Now the Weyl group W of $\mathrm{Spin}(2m)$ consists of transformations of the form $y_k = \epsilon_k x_{\rho(k)}$, where $\epsilon_k = \pm 1$ and $\prod \epsilon_k = +1$ and ρ is a permutation. So W acts transitively on the weights of Δ^+, and thus by a further conjugation, we can take $w = \frac{1}{2}(x_1 + \cdots + x_m)$. Let T_0 be the torus defined by $w = 0$, $T_0 \subset T$.

We claim that T_0 is a maximal torus of G and the normalizer of T_0 in $\mathrm{Spin}(2m)$, $N(T_0)$, is contained in $N(T)$. This is true since if $T_0 \subset T_1$, then $T_0 \subseteq T_1 \cap T$, but for $m > 2$, w is not a multiple of a root, so $T_1 = T$. Similarly, if $gT_0g^{-1} = T_0$, $T_0 \subseteq gTg^{-1} \cap T$ and $gTg^{-1} = T$.

Hence the Weyl group $W(G)$ is contained in the subgroup of W which stabilizes T_0, i.e., transformations of the form $y_k = \epsilon x_{\rho(k)}$, $\epsilon = \pm 1$ which is isomorphic to $S_m \times \mathbb{Z}_2$, where S_m is the symmetric group on m letters. In fact, $W(G) \subset S_m$ since $W(G)$ is generated by reflections in the wall of a Weyl chamber and if $(x, y) \in S_m \times \mathbb{Z}_2$ is of order 2, (x, y) does not leave fixed a hyperplane unless $y = 0$.

We see then that the maximal torus of G is given by $x_1 + \cdots + x_m = 0$ and the Weyl group is contained in the symmetric group on $(x_1, \ldots, x_m)$. But this is the maximal torus and Weyl group of $SU(m)$. $SU(m)$ is simply connected and therefore lifts from $SO(2m)$ to $\mathrm{Spin}(2m)$ where the spin representation Δ^+ restricted to $SU(m)$ is the even part of the complex exterior product representation λ^{even} (see Atiyah, Bott, and Shapiro [10]). Since $\lambda^0 \subset \lambda^{\mathrm{even}}$ and λ^0 is the trivial representation of $SU(m)$, $SU(m) \subset G$. If $SU(m) \neq G_0$, then G would have an extra root but then there would be a point in the interior of a Weyl chamber of $SU(m)$ which was left fixed by an element of $W(G)$. Since $W(G) = W(SU(m))$, this is impossible. Hence $G_0 = SU(m)$ for $m > 2$.

In the case $m = 2$, $\text{Spin}(4) \simeq SU(2) \times SU(2)$ and $\Delta^{\pm}$ are given by projections onto the two factors. The isotropy subgroups are then clearly isomorphic to $SU(2)$.

Since $SO(2m) \subset SO(2m+1)$ have the same maximal torus, we see that the isotropy subgroup for $\text{Spin}(2m+1)$ is $SU(m)$.

Returning to the theorem, we see that if X is a spin manifold with a parallel spinor, then $\pm X^{2m}$ or $\pm X^{2m+1} \times S^1$ admits a reduction of its linear holonomy group to $SU(m)$. The theorem then follows since if $\Phi \subset U(n)$, X is Kähler and if $\Phi \subset SU(n)$, X is Kähler with vanishing Ricci tensor—see Kobayashi and Nomizu [23] and Iwamoto [21].

1.4. Conformal Invariance

Proposition 1.3. *The dimension of the space of harmonic spinors on a manifold X is a conformal invariant.*

Proof. We recall that two metrics g, $\tilde{g}$ are conformally equivalent if there is a C^∞ function σ on X such that $\tilde{g} = e^{2\sigma}g$. Now to compare the Dirac operators corresponding to different metrics, we must first define them on the same vector bundle, so let us fix a *conformal* structure on X, i.e., a reduction of the group of the principal bundle of T from $GL(n, \mathbb{R})$ to $SO(n) \times \mathbb{R}^+$. This defines an isomorphism $T \simeq U \otimes L$, where U is an orthogonal bundle and L is a trivial real line bundle. We take the spinor bundle V corresponding to U.

Given a connection on U, we then have a Dirac operator $P: \Gamma(V) \to \Gamma(V \otimes L^*)$.

A *metric* is now a trivialization of L. If we take the connection on U induced by the riemannian connection on T and use the trivialization of L, then $P: \Gamma(V) \to \Gamma(V)$ is the usual Dirac operator.

If $\tilde{g}$, g are conformally equivalent metrics ($\tilde{g} = e^{2\sigma}g$), then the riemannian connections on T are related by the following formula:

$$\tilde{\nabla}_X Y = \nabla_X Y + (X \cdot \sigma)Y + (Y \cdot \sigma)X - g(X, Y)\,\text{grad}\,\sigma,$$

where X, $Y \in \Gamma(T)$, $(X \cdot \sigma) = \langle d\sigma, X\rangle$, and $g(\text{grad}\,\sigma, Z) = \langle d\sigma, Z\rangle$, where $\langle\ ,\ \rangle$ is the contraction $T^* \otimes T \to \mathbb{R}$.

Fix a local orthonormal basis for U, and let $\{e_i\}$, $\{\tilde{e}_i\}$ be the corresponding orthonormal bases for T relative to the metrics $g, \tilde{g}$. Then $\tilde{e}_i = e^{-\sigma}e_i$. Let $\{\epsilon_i\}$ denote the dual basis of $\{e_i\}$ relative to g.

Then rewriting the above formula in terms of the covariant derivatives D, $\tilde{D}$, we get:

$$\tilde{D}\tilde{e}_i = D(e^{-\sigma}e_i)$$
$$+ e^{-\sigma}\Big(d\sigma \otimes e_i + \langle d\sigma, e_i\rangle \sum_j \epsilon_j \otimes e_j - \sum_j \langle d\sigma, e_j\rangle \epsilon_i \otimes e_j\Big).$$

The connection matrices of the two induced connections on U are then related by:

$$\tilde{\omega}_{ij} = \omega_{ij} + \epsilon_j\langle d\sigma, e_i\rangle - \epsilon_i\langle d\sigma, e_j\rangle.$$

Consider now the two Dirac operators P, $\tilde{P}$: $\Gamma(V) \to \Gamma(V \otimes L^*)$. We compose with the isomorphism $\Gamma(V \otimes L^*) \to^{\varphi} \Gamma(V)$ defined by the metric g and compute the action of P, $\tilde{P}$ on an element ψ of the local spinor basis.

$$\varphi\tilde{P}\psi = \varphi P\psi + \tfrac{1}{4}\Big(\sum_{i,j} e_j e_i e_j\langle d\sigma, e_i\rangle - e_j e_i e_j\langle d\sigma, e_j\rangle\Big)\psi$$
$$= \varphi P\psi + \tfrac{1}{4}\Big(\sum_{i,j} e_i\langle d\sigma, e_i\rangle - 2\delta_{ij}e_j\langle d\sigma, e_i\rangle + e_j\langle d\sigma, e_j\rangle\Big)\psi$$
$$= \varphi P\psi + \tfrac{1}{2}(n-1)\, d\sigma \cdot \psi.$$

Since P, $\tilde{P}$ have the same symbol and the endomorphism $d\sigma$ is globally defined, then $\tilde{P}\psi = P\psi + \frac{1}{2}(n-1)\, d\sigma \cdot \psi$ for any spinor ψ.

Note that $e^{-\sigma}P(e^{\sigma}\psi) = P\psi + d\sigma \cdot \psi$. Hence,

$$\tilde{P}\psi = e^{\frac{-(n-1)\sigma}{2}} P\Big(e^{\frac{(n-1)\sigma}{2}}\psi\Big),$$

and so if $\tilde{P}\psi = 0$, then $P(e((n-1)\sigma/2)\psi) = 0$, i.e., the dimension of the space of harmonic spinors is a conformal invariant.

Remarks. (1) Let us define a spin representation for the conformal group by $\tau(g, \lambda) = \rho(g)\lambda^{((n-1)/2)}$, where $(g, \lambda) \in \mathrm{Spin}(n) \times \mathbb{R}^+$ and ρ is the usual spin representation. Let $\tilde{V}$ be the associated vector bundle, then a metric defines an isomorphism φ: $\tilde{V} \cong V$. We define the Dirac operator $\tilde{P}$: $\Gamma(\tilde{V}) \to \Gamma(\tilde{V} \otimes L^*)$ by $\varphi^{-1}P\varphi$. Then the proof of Proposition

1.3 shows that $\tilde{P}$ is independent of the choice of metric in the conformal class, so we have a canonical Dirac operator associated to the conformal structure.

(2) On S^1, conformal invariance trivially implies that dim H is independent of the metric. Since $H^1(S^1, \mathbb{Z}_2) = \mathbb{Z}_2$, there are two spin structures. The two spinor bundles have real structures and are the trivial line bundle and the Hopf bundle. On the trivial bundle, dim $H = 1$; on the Hopf bundle $H = 0$.

(3) Since $GL(1, \mathbb{C}) \cong \mathbb{C}^* \cong SO(2) \times \mathbb{R}^+$, on a two-dimensional manifold, Proposition 1.3 implies that dim H depends only on the complex structure. We shall see this more generally in Sections 2.1–2.4.

(4) We mention here the Künneth formula for the tensor product of elliptic complexes (see Atiyah and Bott [9]). If X and Y are two spin manifolds and we take the product metric on $X \times Y$, then the dimension of the space of harmonic spinors on the product (h) is related to the dimensions of the space of harmonic spinors on the factors (h_1, h_2) via the Künneth formula by the following:

$$\begin{array}{ll} \text{even} \times \text{even} & \begin{array}{l} h^+ = h_1^+ h_2^+ + h_1^- h_2^- \\ h^- = h_1^+ h_2^- + h_1^- h_2^+ \end{array} \\ \text{even} \times \text{odd} & h = (h_1^+ + h_1^-)\, h_2 \\ \text{odd} \times \text{odd} & h^+ = h^- = h_1 h_2 . \end{array}$$

In particular, for the flat torus $T^n = S^1 \times \cdots \times S^1$, of the 2^n spin structures, only the one corresponding to the trivial lifting admits a harmonic spinor.

2.1. Harmonic Spinors on a Kähler Manifold

Let X be a complex manifold; then the lifting

$$\begin{array}{ccc} & \overset{l}{\nearrow} & \mathrm{Spin}^c(2n) \\ & & \downarrow \\ U(n) & \hookrightarrow & SO(2n) \end{array} \qquad \text{(see [10])}$$

defines a canonical Spin^c structure on X.

THEOREM 2.1. *Let X be a Kähler manifold; then with respect to the canonical Spin^c structure,*

$$H^+ \cong H^{\text{even}}(X, \mathcal{O}),$$
$$H^- \cong H^{\text{odd}}(X, \mathcal{O}),$$

where $\mathcal{O}$ denotes the sheaf of germs of local holomorphic functions on X.

Proof. The Spin^c representation restricted to $U(n)$ is the exterior product representation, with Clifford multiplication given by the following:

$$\mathbb{C}^n \otimes_{\mathbb{R}} \lambda^*\mathbb{C}^n \to \lambda^*\mathbb{C}^n$$
$$v \otimes w \mapsto d(v)w - \delta(v)w,$$

where $d(v)w = v \wedge w$ and $\delta(v)$ is its adjoint relative to the hermitian structure. The $\mathbb{Z}_2$-grading is given by the even-odd decomposition of the exterior algebra (see [10]).

Let V be a complex vector space with hermitian form H; then as usual we have a complex linear embedding $V \subset V^* \otimes_{\mathbb{R}} \mathbb{C}$ (where V^* is the real dual of V) given by $v \mapsto \varphi(v) + i\varphi(iv)$, where $\varphi: V \to V^*$ is the isomorphism defined by the bilinear form B given by the real part of H. B induces a hermitian form $\tilde{H}$ on $V^* \otimes_{\mathbb{R}} \mathbb{C}$ and hence on V.

$$\begin{aligned}
&\tilde{H}(\varphi(v) + i\varphi(iv), \varphi(w) + i\varphi(iw)) \\
&\quad = B(\varphi(v), \varphi(w)) + B(\varphi(iv), \varphi(iw)) + iB(\varphi(iv), \varphi(w)) - iB(\varphi(v), \varphi(iw)) \\
&\quad = 2(B(v, w) + iB(iv, w)) = 2H(v, w).
\end{aligned}$$

So the induced hermitian form on $V \subset V^* \otimes_{\mathbb{R}} \mathbb{C}$ is twice the original form.

On the manifold X we have a complex linear isomorphism $\psi: T \cong T^{0,1}$ between the tangent bundle and the bundle of $(0, 1)$ forms such that $\langle\psi(X), \psi(Y)\rangle = 2\langle X, Y\rangle$. We can thus identify the bundle of spinors with $\lambda^* T^{0,1}$ and define Clifford multiplication by $a \in T^*$ as $\sqrt{2}(d(\alpha^{0,1}) - \delta(\alpha^{0,1})$, where $\alpha^{0,1}$ is the $(0, 1)$ component of α.

We claim the Dirac operator $P = \sqrt{2}(\bar{\partial} + \bar{\partial}^*)$, where $\bar{\partial}: \Gamma(T^{0,p}) \to \Gamma(T^{0,p+1})$, is the usual exterior derivative in the Dolbeault complex.

Given a connection D on a vector bundle E, any first-order linear differential operator $P: \Gamma(E) \to \Gamma(F)$ may be written uniquely in the form $P = \sigma D + \tau$, where $\sigma: \Gamma(E \otimes T^*) \to \Gamma(F)$ is the symbol and $\tau \in \Gamma(\text{Hom}(E, F))$.

Take the riemannian connection on the spinor bundle $V(\cong\lambda^* T^{0,1})$; then the Dirac operator $P = \sigma D$, where σ is Clifford multiplication. The symbol of $\bar\partial + \bar\partial^*$ is $d(\alpha^{0,1}) - \delta(\alpha^{0,1})$, so P and $\sqrt{2}(\bar\partial + \bar\partial^*)$ have the same *symbol*. It remains to show that the zero-order term $\tau(\bar\partial + \bar\partial^*) = 0$ relative to the riemannian connection. Since the volume form is parallel $\tau(\bar\partial^*) = \tau(\bar\partial)^*$, we need only prove $\tau(\bar\partial) = 0$. Now $d = \partial + \bar\partial$, where $d\colon \Gamma(\lambda^p T^* \otimes_{\mathbb{R}} \mathbb{C}) \to \Gamma(\lambda^{p+1} T^* \otimes_{\mathbb{R}} \mathbb{C})$ is the exterior derivative and $\tau(d) = 0$ since the riemannian connection has no torsion. On a *Kähler* manifold, the riemannian connection D commutes with the almost complex structure J and so $\tau(\bar\partial) = 0$.

The theorem follows from the Hodge theory of the Dolbeault complex:

$$\cdots \to \Gamma(T^{0,p}) \xrightarrow[\bar\partial]{} \Gamma(T^{0,p+1}) \to \cdots .$$

THEOREM 2.2. *Let X be a compact Kähler manifold; then*

(1) *X is spin iff the canonical bundle K has a square root* (i.e., *a complex line bundle L such that $L \otimes L \cong K$*);

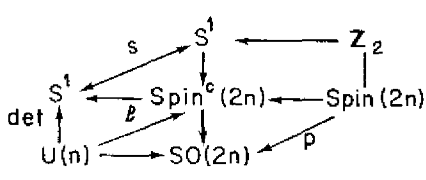
(2) *there is a one-to-one correspondence between spin structures on X and holomorphic square roots of K;*

(3) *under this correspondence,*

$$H^+ \cong H^{\text{even}}(X, \mathcal{O}(L)),$$

$$H^- \cong H^{\text{odd}}(X, \mathcal{O}(L)).$$

Proof. (1) We have the following commutative diagram of group homomorphisms:



where $s(x) = x^2$.

If $u \in U(n) \subset SO(2n)$, then $p^{-1}(u) = \pm\ell(u) \det u^{-1/2} \in \mathrm{Spin}(2n)$. Hence the lifting of a cocycle $u_{\alpha\beta}$ to a Spin$(2n)$ cocycle corresponds bijectively to the lifting of the S^1-cocycle $\det u_{\alpha\beta}^{-1}$ to an S^1-cocycle $h_{\alpha\beta}$ such that $h_{\alpha\beta}^2 = \det u_{\alpha\beta}^{-1}$. Since $\det u_{\alpha\beta}^{-1}$ represents the canonical bundle K, X is spin iff K has a square root.

(2) Let $\mathcal{O}^*$ denote the sheaf of germs of nonvanishing local holomorphic functions on X. Then we have an exact sequence of sheaves:

$$1 \to \mathbb{Z}_2 \to \mathcal{O}^* \to \mathcal{O}^* \to 1$$
$$x \mapsto x^2.$$

In the corresponding exact cohomology sequence, we have:

$$H^1(X, \mathbb{Z}_2) \xrightarrow{\alpha} H^1(X, \mathcal{O}^*) \to H^1(X, \mathcal{O}^*) \xrightarrow{\beta} H^2(X, \mathbb{Z}_2) \to,$$

where α is an injection for compact X.

A holomorphic line bundle $L \in H^1(X, \mathcal{O}^*)$ thus has a holomorphic square root iff the topological obstruction $\beta(L) \in H^2(X, \mathbb{Z}_2)$ is zero (in fact $\beta(L) = c_1(L) \bmod 2$). Hence X is spin iff K has a *holomorphic* square root. From the first part of the proof, two liftings of an S^1-cocycle to the double covering differ by a $\mathbb{Z}_2$-cocycle: since α is injective, the cohomology class of this cocycle distinguishes between holomorphically distinct square roots of K and so we get a one-to-one correspondence between spin structures and holomorphic square roots of K (see also Atiyah [8]).

(3) The spin representation takes $\ell(u) \det u^{-1/2}$ into $\lambda^*(u) \otimes (\det u)^{-1/2}$ and so the bundle of spinors on a Kähler manifold is isomorphic to $\lambda^* T^{0,1} \otimes L$, where L is a square root of K. As in Theorem 2.1, we show that $P = \sqrt{2}(\bar\partial + \bar\partial^*)$, where $\bar\partial(\psi \otimes s) = \bar\partial\psi \otimes s$ if s is a local holomorphic section of L; i.e., $\bar\partial$ is the coboundary operator in the Dolbeault complex of L.

The symbols of the two operators are the same, so again we must show that $\bar\partial$ factors through the connection induced on $\lambda^* T^{0,1} \otimes L$ via the riemannian connection. We showed this for $\lambda^* T^{0,1}$, so it remains to show that if s is a local holomorphic section of L, then $Ds \in \Gamma(T^{1,0} \otimes L)$ for then $\bar\partial(\psi \otimes s) = \sigma D(\psi \otimes s)$. $L \otimes L \simeq K$ and L has the connection induced from K, so it suffices to prove the above statement for K. But $dz_1 \wedge \cdots \wedge dz_n$ is a local holomorphic section of K and $D(dz_i) \in \Gamma(T^{1,0} \otimes T^{1,0})$ since D has no torsion and so the skew part of $D(dz_i)$ is $d(dz_i)$, which is zero. Hence $D(dz_1 \wedge \cdots \wedge dz_n) \in \Gamma(T^{1,0} \otimes K)$.

(4) We sometimes need to consider spinors with coefficients in a vector bundle E with connection. We then have a connection on $V \otimes E$ and a Clifford multiplication on the left, so that we can define a Dirac operator as in Sections 1.1–1.4. Suppose now X is a Kähler manifold and E is a holomorphic hermitian vector bundle. If we choose the unique unitary connection on E such that $Dv = \sum \omega_i \otimes v_i$, where

$\{v_i\}$ is a local holomorphic basis and the ω_i are (1, 0) forms, and construct the Dirac operator, we see as in the above argument that $P = \sqrt{2}(\bar\partial + \bar\partial^*)$, and, in particular, we can identify the harmonic spinors with coefficients in E with $H^*(X, \mathcal{O}(L \otimes E))$.

In Sections 2.1–2.4 we shall adopt the usual convention of writing line bundles additively, i.e., $L \otimes M = L + M$. We shall use Theorem 2.2 to compute the dimension of the space of harmonic spinors for particular Kähler manifolds but first we make some remarks:

Remarks. (1) Since $H^p(X, \mathcal{O}(\frac{1}{2}K))$ is defined entirely in terms of the *complex structure* on X, dim H is independent of the choice of Kähler metric defining the same complex structure. In real dimension 2, since $\mathbb{C}^* \cong SO(2) \times \mathbb{R}^+$ via $z \mapsto (z/|z|) \cdot |z|$, this is equivalent to saying that dim H is a conformal invariant which we have seen in Sections 1.1–1.4.

(2) Serre duality asserts that if L is a holomorphic line bundle, $H^p(X, \mathcal{O}(L)) \cong H^{n-p}(X, \mathcal{O}(K - L))$. Hence, if $L = \frac{1}{2}K$, we have the duality:

$$H^p(X, \mathcal{O}(\tfrac{1}{2}K)) \cong H^{n-p}(X, \mathcal{O}(\tfrac{1}{2}K)).$$

(3) If we take the canonical Spin^c structure on a Kähler manifold with positive definite Ricci tensor, then the endomorphism Ω in Theorem 1.1. is positive on $\bigotimes_{p=1}^{n} \lambda^p T^{0,1}$ and zero on $\lambda^0 T^{0,1}$. Combined with Theorem 2.1, this yields Bochner's vanishing theorem, i.e., a Kähler manifold with positive definite Ricci tensor admits no holomorphic p-forms for $p > 0$. This is a special case of Kodaira's vanishing theorem, but Kodaira's theorem does not appear to be more powerful than Lichnerowicz's in general. For example, one can show by Kodaira's method that if the scalar curvature is positive, then $H^0(X, \mathcal{O}(\frac{1}{2}K))$ (and hence by duality $H^n(X, \mathcal{O}(\frac{1}{2}K))$) vanishes (see Kobayashi and Wu [24]), but it is not clear that one can deduce $H^p(X, \mathcal{O}(\frac{1}{2}K)) = 0$ for $0 < p < n$ which is what Lichnerowicz's theorem gives.

(4) Let X be a compact complex manifold and L_1, L_2 holomorphic line bundles on X. Then we have a bilinear map

$$m: H^0(X, \mathcal{O}(L_1)) \times H^0(X, \mathcal{O}(L_2)) \to H^0(X, \mathcal{O}(L_1 + L_2))$$

defined by multiplication, i.e., if s_1, s_2 are holomorphic sections of L_1, L_2, then

$$m(s_1, s_2) = s_1 s_2,$$

m induces a corresponding differentiable map $\tilde{m}$ on the projective spaces $P(L) = \mathbb{P}(H^0(X, \mathcal{O}(L)))$,

$$\tilde{m}: P(L_1) \times P(L_2) \to P(L_1 + L_2).$$

The points of $P(L)$ correspond to effective divisors of L, i.e., the zeros of holomorphic sections of L. Let D be a divisor of $L_1 + L_2$; then D is in the image of $\tilde{m}$ iff $D = D_1 + D_2$, where D_1 and D_2 are effective divisors of L_1 and L_2, respectively. Since D has only a finite number of irreducible components, $D = D_1 + D_2$ in only a finite number of ways, i.e., $\tilde{m}^{-1}(pt)$ is finite. Hence $\dim P(L_1 + L_2) \geqslant \dim P(L_1) + \dim P(L_2)$.

Consider $L_1 = L_2 = \frac{1}{2}K$. Then if $h^0 = \dim H^0(X, \mathcal{O}(\frac{1}{2}K))$ and p_g is the geometric genus $= \dim H^0(X, \mathcal{O}(K))$, we have $(p_g - 1) \geqslant 2(h^0 - 1)$, i.e.,

$$h^0 \leqslant \left[\frac{p_g + 1}{2}\right],$$

where $[x]$ denotes the integer part of x. This gives an upper bound on h^0 which as we shall see is sometimes attained.

2.2. Riemann Surfaces

Every oriented two-dimensional manifold X is a spin manifold since $w_2(X) = 0$. Furthermore, since $SO(2) \cong U(1)$, every riemannian metric on X is a Kähler metric, so we lose no generality by considering Kähler metrics. By Serre duality, $H^0(X, \mathcal{O}(\frac{1}{2}K)) \cong H^1(X, \mathcal{O}(\frac{1}{2}K))$, so we need only compute h^0 to find $\dim H$. Hence the dimension of the space of harmonic spinors on a 2-manifold is independent of the metric iff h^0 is independent of the complex structure.

We note that if X is of genus g, there are $\# H^1(X, \mathbb{Z}_2) = 2^{2g}$ different spin structures and that $p_g = g$.

Proposition 2.3. *If $g < 3$, the dimension of the space of harmonic spinors is independent of the metric.*

Proof. From Remark 4 above, we have

$$h^0 \leqslant \left[\frac{g + 1}{2}\right].$$

Hence, if $g = 0$, $h^0 = 0$, and if $g < 3$, $h^0 = 0$ or 1. Thus we can find the number of square roots of K with no holomorphic sections by considering the number for which h^0 is even, and it is classically known that there are $2^{g-1}(2^g + 1)$ such square roots (see Atiyah [8]). So for $g = 1$, 3 square roots of the canonical bundle have no holomorphic sections; one (the trivial one) has one section. For $g = 2$, there are 10 square roots with no holomorphic sections and 6 with one.

PROPOSITION 2.4. *If X is hyperelliptic, $h^0 = [(g + 1)/2]$ for some square root of K. Moreover, if g is even, there are at least $2(g + 1)$ such square roots.*

Proof. We refer to Gunning [20] for terminology and basic facts about the Weierstrass gap sequence.

Let $p \in X$, and let $\gamma(\nu p)$ denote the dimension of the space of holomorphic sections of the line bundle defined by the divisor νp, ν being a positive integer.

Then

$$\gamma(\nu p) = \nu + 1 - \{\#\text{ gaps} \leqslant \nu \text{ in Weierstrass gap sequence at } p\}$$

(see Theorem 14 in [20]). If p is a hyperelliptic Weierstrass point, the gap sequence is

$$1 < 3 < 5 \cdots \qquad \cdots < 2g - 1.$$

Hence $\gamma((2g - 2)p) = 2g - 1 - (g - 1) = g$, and so $(2g - 2)p$ defines a line bundle with first Chern class $(2g - 2)[X]$ and g holomorphic sections which must therefore be the canonical bundle K. $(g - 1)p$ then defines a square root of K and

$$\gamma((g - 1)p) = (g - 1) + 1 - \begin{cases} g/2 & g \text{ even}, \\ (g - 1)/2 & g \text{ odd}, \end{cases}$$

i.e.,

$$\gamma((g - 1)p) = \left.\begin{matrix} g/2 & g \text{ even} \\ (g + 1)/2 & g \text{ odd} \end{matrix}\right\} = \left[\frac{g + 1}{2}\right].$$

Our square roots of the canonical bundle having $[(g + 1)/2]$ holomorphic sections are equivalent as divisors to $(g - 1)p$, where p is a Weierstrass point. There are $2(g + 1)$ Weierstrass points p_i on a hyperelliptic surface and these are the branch points of a ramified double covering $f: X \to \mathbb{P}^1$. Since all points are equivalent as divisors on $\mathbb{P}^1$, on X we have $2p_i \sim 2p_j$. Hence if g is even, $(g - 1)p_i \sim (g - 1)p_j$ implies $p_i \sim p_j$. But from the Weierstrass gap sequence, $\gamma(p_i) = 1$ and so

$p_i \sim p_j$ implies $p_i = p_j$, thus we have $2(g+1)$ *distinct* square roots with $h^0 = [(g+1)/2]$.

A partial converse to the above is provided by a theorem of H. Martens ([26 Theorem 3.1]), which we state as follows:

PROPOSITION 2.5 (Martens). *If* $h^0 = [(g+1)/2]$, *then* X *is one of the following types*:

(a) *hyperelliptic,*

(b) $g = 4$,

(c) $g = 6$.

In the nonhyperelliptic cases of $g = 4$ and 6, there is only one square root having $[(g+1)/2]$ holomorphic sections (by a result of Farkas [19]), but from Proposition 2.4 a hyperelliptic surface of genus 4 has at least 10 such square roots and genus 6 at least 14. We may therefore say that hyperelliptic surfaces are distinguished by the property of having the maximal number of harmonic spinors for the maximal number of spin structures. Since for $g \geqslant 3$ there exist hyperelliptic and nonhyperelliptic surfaces, we may state here the main result in 2 dimensions:

THEOREM 2.6. *The dimension of the space of harmonic spinors on a two-dimensional riemannian manifold varies with the choice of metric.*

By taking products of Riemann surface and using the Künneth formula, we can construct manifolds in every dimension on which dim H depends upon the metric, but these all have several spin structures. We now look at simply connected manifolds where, since $H^1(X, \mathbb{Z}_2) = 0$, there is a unique spin structure.

3. Algebraic Surfaces

Every nonsingular projective algebraic variety is a Kähler manifold, so we can apply Theorem 2.2 to an algebraic surface with $w_2(X) = 0$. We put $h^p = \dim H^p(X, \mathcal{O}(\frac{1}{2}K))$, then $h^0 = h^2$ by Serre duality and $h^0 - h^1 + h^2 = \hat{A}(X)$ by the Riemann–Roch theorem, hence we need only calculate h^0.

By Remark 4 in **2.1** we have:

$$h^0 \leqslant \frac{p_g + 1}{2}. \tag{1}$$

By the Riemann–Roch theorem applied to the trivial line bundle, we have:

$$1 - q + p_g = (c_1^2 + c_2)/12 = \text{Todd genus}, \tag{2}$$

where q is the irregularity $= \dim H^1(X, \mathcal{O})$. Also by the Riemann–Roch theorem applied to the holomorphic line bundle $\frac{1}{2}K$, we have

$$h^0 - h^1 + h^0 = (c_1^2 + c_2)/12 - c_1^2/8 = \hat{A} \text{ genus}. \tag{3}$$

Hence from (1), (2), and (3), we get

$$(c_1^2 + c_2)/12 - c_1^2/8 \leqslant 2h^0 \leqslant (c_1^2 + c_2)/12 + q. \tag{4}$$

Remarks. (1) It follows from the inequality (4) that if X is a spin algebraic surface, $q \geqslant -c_1^2/8$. Now suppose the intersection matrix is of type (r, s), then $r + s = b_2$ and $r - s = \operatorname{sign} X = (c_1^2 - 2c_2)/3$. Also, $c_2 =$ Euler characteristic $= 2 - 4q + b_2$. We can therefore express the inequality as:

$$c_2 - b_2 - 2 \leqslant c_1^2/2,$$

i.e,

$$3 \operatorname{sign} X \geqslant -2(b_2 + 2),$$

i.e.,

$$3(r - s) \geqslant -2(r + s + 2),$$

i.e.,

$$5r - s + 4 \geqslant 0.$$

This is a topological condition that a four-dimensional spin manifold must satisfy in order to be algebraic.

(2) Suppose $\pi_1(X) = \{1\}$, then $q = 0$. If $c_1^2 = 0$, then from the inequality (4), $h^0 = c_2/24$ and $h^1 = 0$, hence in this case h^p is a topological invariant for Kähler metrics defining algebraic structures.

(3) Every spin surface X is a relatively minimal model, for suppose E is the divisor of an exceptional curve of the first kind. Then E is nonsingular, rational, and of self-intersection $E \cdot E = -1$. Therefore, from the formula (for a nonsingular divisor D) $D \cdot (D + K) = 2(\pi(D) - 1)$, we have $K \cdot E = -1$. However, the spin condition implies that $K = 2F$ for some divisor F, so $K \cdot E$ is an even number. Hence, X has no exceptional curves of the first kind and is thus a relatively minimal model.

(4) Applying Remarks 2 and 3 to Enriques' classification of algebraic surfaces (see Shafarevitch [30]), we see that $h^1 = 0$ (and hence h^p is independent of the algebraic structure) for all simply connected algebraic surfaces except the class $\kappa = 2$ (i.e., surfaces of general type) and possibly rational surfaces. We shall see now that this is true for rational surfaces and a considerable number of surfaces of general type.

(a) *Rational surfaces.* Every relatively minimal model of a rational surface (except $\mathbb{P}^2$ which is not spin) is a fibre bundle over $\mathbb{P}^1$ with fibre $\mathbb{P}^1$ and hence in particular has sign $X = 0$, and therefore $\hat{A}(X) = 0$. Hence, $h^1 = 2h^0$. But $h^0 \leqslant (p_g + 1)/2$ and $p_g = 0$ for a rational surface, so $h^0 = h^1 = h^2 = 0$.

(b) *Complete intersections.* We consider the algebraic surface $V_2(a_1, ..., a_r)$ given by the intersection of r nonsingular hypersurfaces $F(a_1), ..., F(a_r)$ of degrees $a_1, ..., a_r$ in $\mathbb{P}^{r+2}$ in general position. From the Lefschetz theorem on hyperplane sections, such a variety is simply connected.

PROPOSITION. *$V_2(a_1, ..., a_r)$ is spin iff $\sum_1^r a_i - (r + 3)$ is even.*

Proof. The total Chern class of V is given by

$$c(V) = (1 + [H])^{r+3} (1 + a_1[H])^{-1} \cdots (1 + a_r[H])^{-1},$$

where $[H] \in H^2(V, \mathbb{Z})$ is the cohomology class given by a hyperplane section H in $\mathbb{P}^{r+2}$. Thus

$$c_1(V) = \left((r + 3) - \sum a_i\right) [H],$$

so if $\sum a_i - (r + 3)$ is even, $w_2(V) = c_1(V) \bmod 2 = 0$ and V is spin.

The converse will follow if we show that H is primitive, i.e., $[H] \neq mD$ for any $D \in H^2(V, \mathbb{Z})$. For a complete intersection of dimension >2, the Lefschetz theorem says that $[H]$ generates $H^2(V, \mathbb{Z})$. Let $V_2 = V_3 \cap F$ and consider the exact cohomology sequence:

$$\cdots \to H^2(V_3) \xrightarrow{j^*} H^2(V_3 \cap F) \to H^3(V_3, V_3 \cap F) \to \cdots .$$

If $j^*[H]$ is not primitive, then since $[H]$ generates $H^2(V_3)$, there will be torsion in $H^3(V_3, V_3 \cap F)$ and hence in $H_2(V_3, V_3 \cap F)$. Consider now the exact homology sequence:

$$\cdots \to H_2(V_3 \cap F) \xrightarrow{j_*} H_2(V_3) \to H_2(V_3, V_3 \cap F) \xrightarrow{i} H_1(V_3 \cap F) \to \cdots .$$

j_* is surjective by the Lefschetz theorem, hence i is injective. But $H_1(V_3 \cap F) = 0$ since V_2 is simply connected, so there is no torsion in $H_2(V_3, V_3 \cap F)$ and $[H]$ is primitive.

The canonical bundle of V_2 is thus given by $K = (\sum a_i - (r+3))H$, and if $\sum a_i \neq r+3$, then $c_1^2 > 0$, so V_2 is a surface of general type or rational.

If $\sum a_i - (r+3) < 0$, then $a_i = 1$ for $i \neq 1$ (say) and $a_1 = 2$, i.e., V_2 is a quadric in $\mathbb{P}^3$, which is rational and which we have therefore already considered in (a).

If $\sum a_i - (r+3) = 2s (s > 0)$, then the unique square root of K is sH. It is well-known, however, that $H^1(V_2, \mathcal{O}(sH)) = 0$ for a complete intersection.

Hence $h^1 = 0$.

(c) *Ramified coverings.* In dimension 1, the most interesting varieties from our point of view were hyperelliptic curves, i.e., ramified double coverings of $\mathbb{P}^1$. We now consider a two-dimensional analog: cyclic coverings of $\mathbb{P}^2$ branched over a nonsingular curve.

Let $C \subset \mathbb{P}^2$ be a nonsingular curve of degree pq. Then we can construct the p-fold covering $f: X \to \mathbb{P}^2$ ramified over the branch curve C. Let $C' = f^{-1}(C)$.

PROPOSITION. *X is simply connected.*

Proof. Let $D(N)$, $S(N)$ (resp. $D(N')$, $S(N')$) be the disc and sphere bundles of the normal bundle N (resp. N') of C (resp. C') in $\mathbb{P}^2$ (resp. X). Then $\pi_1(\mathbb{P}^2 - D(N)) \cong \mathbb{Z}_{pq}$ and the generator is given by the inclusion i of a fibre of $S(N)$:

$$\pi_1(S^1) \xrightarrow{i} \pi_1(S(N)) \xrightarrow{j} \pi_1(\mathbb{P}^2 - D(N))$$

(see Zariski [31, Chapter VIII]).

Now $X - \mathring{D}(N')$ is a p-fold unramified covering of $\mathbb{P}^2 - \mathring{D}(N)$, so $\pi_1(X - \mathring{D}(N')) \cong \mathbb{Z}_q$ with generator given by the inclusion of a fibre in $S(N')$. Let $z \in \pi_1(S(N'))$ be this generator.

We claim that the homomorphism

$$\pi_1(S(N')) \xrightarrow{i_1 \times i_2} \pi_1(X - \mathring{D}(N')) \times \pi_1(D(N'))$$

is surjective (where i_1, i_2 are induced by the natural inclusions). This is true since i_2 is surjective from the exact sequence of the fibration $S^1 \to S(N') \to C'$, so given $(j'(z^m), w)$ on the right-hand side, take $w' \in \pi_1(S(N'))$

s.t. $i_2(w') = w$. If $i_1(w') = j'(z^n)$, then $i_1 \times i_2(z^{m-n}w') = (j'(z^m), w)$, so $i_1 \times i_2$ is surjective.

Hence the subgroup generated by $\pi_1(S(N'))$ in the free product of $\pi_1(X - \mathring{D}(N'))$ and $\pi_1(D(N'))$ is the whole group and by Van Kampen's theorem, $\pi_1(X) = \{1\}$.

PROPOSITION. *X is spin iff p is even and q is odd.*

Proof. Let $f: X \to \mathbb{P}^2$ be the projection, then the derivative of f defines a natural homomorphism of sheaves:

$$f^*: \mathcal{O}(f^*(K_{\mathbb{P}^2})) \to \mathcal{O}(K_X)$$

(where K_X is the canonical bundle of X), or, in other words, a holomorphic section of $K_X - f^*K_{\mathbb{P}^2}$. If α is a local nonvanishing holomorphic n-form on $\mathbb{P}^2$, then $f^*\alpha$ vanishes to order $(p - 1)$ on the branch locus C'. Hence,

$$K_X - f^*K_{\mathbb{P}^2} = (p - 1)\, C',$$

where C' denotes the line bundle of the divisor C'. Hence,

$$K_X = f^*((p - 1)\, qH - 3H),$$

where H is the line bundle on $\mathbb{P}^2$ defined by a hyperplane section. Thus if p is even and q odd, $c_1(X) = 0 \bmod 2$ and X is spin.

The converse will follow if we can show that $f^*([H]) \in H^2(X, \mathbb{Z})$ is primitive. Now the pq-fold ramified covering is a nonsingular hypersurface Y in $\mathbb{P}^3$ given by the equation

$$x_0^{pq} + g(x_1, x_2, x_3) = 0,$$

where the polynomial g defines the curve C in $\mathbb{P}^2$. $\mathbb{Z}_{pq}$ acts on Y via the action on $\mathbb{P}^3$ given by

$$n \cdot (x_0, x_1, x_2, x_3) = (\exp(2\pi ni/pq)\, x_0, x_1, x_2, x_3)$$

and defines the projection $Y \to^{f'} X \to^{f} \mathbb{P}^2$. The divisor of the branch curve C'' on Y is equivalent to $(ff')^*H$ but this is also the divisor given by $x_0 = 0$, i.e., a hyperplane section H' in $\mathbb{P}^3$. Hence $H' = (ff')^*H$ as holomorphic line bundles. We know that $[H']$ is primitive on Y, hence $(f')^*[H]$ must be primitive on X.

Hence if X is spin, $((p-1)q-3)=2s$, and the unique square root of K is given by $\frac{1}{2}K=sf^*H$.

If $p=2$, $q=1$, then X is a quadric and therefore rational. If $p=2$, $q=3$, then $K=0$. Otherwise, $s>0$.

Now if G is a finite group of automorphisms of a complex manifold X and W is a holomorphic vector bundle on X/G, $f: X\to X/G$ the projection map, then

$$(H^p(X,\mathcal{O}(f^*W)))^G\cong H^p(X/G,\mathcal{O}(W)),$$

where V^G denotes the fixed part under the action of G (see Atiyah and Segal [12]).

Taking $X=Y$, $G=\mathbb{Z}_q$, $W=\frac{1}{2}K$, then $f^*\frac{1}{2}K=sH'$ and $H^1(Y,\mathcal{O}(sH'))=0$ since Y is a hypersurface, hence $H^1(X,\mathcal{O}(\frac{1}{2}K))=0$, i.e., $h^1=0$.

Remark. In view of the preceding (somewhat restricted) examples, it is tempting to *conjecture* the following: Let X be a simply connected algebraic spin surface; then for a generic complex structure, $H^1(X,\mathcal{O}(\frac{1}{2}K))=0$.

If $h^1=0$, then $\hat{A}(X)=2h^0\geqslant 0$, and hence $\operatorname{sign} X\leqslant 0$. This is Zappa's conjecture, which is known to be false, but the counterexamples (see Atiyah [7], Borel [15]) are not simply connected—in fact they are $K(\pi,1)$'s.

2.4. Birational Invariance

We have seen that in certain cases, the dimension of the space of harmonic spinors on an algebraic variety X depends upon the complex structure: it is natural to ask whether it is invariant under the algebraic notion of birational equivalence. We know for example that $\dim H^p(X,\mathcal{O}(K))$ and the plurigenera $\dim H^0(X,\mathcal{O}(mK))$ $(m>0)$ *are* birational invariants. We ask now whether $\dim H^p(X,\mathcal{O}(\frac{1}{2}K))$ is invariant.

In dimension 1, birational equivalence implies biholomorphic equivalence and so the invariance is trivial. In dimension 2, we saw in Remark 3 of **2.3** that a spin manifold was a minimal model. Except for ruled surfaces, a minimal model is unique up to biholomorphic equivalence and so $\dim H^p(X,\mathcal{O}(\frac{1}{2}K))$ is again invariant. For a ruled surface, every minimal model (except $\mathbb{P}^2$) is a fibre bundle with base a curve B and fibre $\mathbb{P}^1$ and thus has zero signature; also, $p_g=0$, so by the same argument as for rational surfaces, there are no harmonic spinors. Hence in dimension 2, $\dim H^p(X, \mathcal{O}(\frac{1}{2}K))$ is a birational invariant.

We consider the question of invariance in higher dimensions under the birational equivalence of "blowing up."

THEOREM 2.5. *Let X' be obtained from X by blowing up a subvariety $Y \subset X$; then*

(1) *X' is spin iff X is spin and codim Y is odd*;

(2) *If X' and X are spin, the projection $f: X' \to X$ induces a one-to-one correspondence between the spin structures on X and those on X'*;

(3) *under this correspondence,*

$$\dim H^p(X', \mathcal{O}(\tfrac{1}{2}K')) = \dim H^p(X, \mathcal{O}(\tfrac{1}{2}K)).$$

Proof. For details on blowing up, we refer to Porteous [28].

(1) Let $Y \subset X$ be of codimension m, with normal bundle N. Then $f^{-1}(Y)$ is the codimension 1 subvariety $Y' \simeq \mathbb{P}(N)$. If α is a local nonvanishing holomorphic n-form on X, then $f^*\alpha$ vanishes to order $(m-1)$ on Y' and

$$K_{X'} = f^*K_X + (m-1)E,$$

where E is the line bundle defined by the divisor Y'. Hence $c_1(X') = f^*c_1(X) + (m-1)[E]$, and if X is spin and m is odd, $c_1(X') \equiv 0 \bmod 2$ and so X' is spin. Conversely, we have the split exact sequence

$$0 \to H^2(X, \mathbb{Z}) \xrightarrow{f^*} H^2(X', \mathbb{Z}) \leftrightarrows \mathbb{Z} \to 0,$$

where the splitting is defined by $1 \mapsto [E]$. Hence if $c_1(X') = f^*c_1(X) + (m-1)[E] \equiv 0 \bmod 2$, then m must be odd and $c_1(X) \equiv 0 \bmod 2$.

(2) We see from above that if $\frac{1}{2}K_X$ is a holomorphic square root of K_X, then $f^*\frac{1}{2}K_X + (m-1)/2\,[E]$ is a holomorphic square root of $K_{X'}$. Since f^* induces an isomorphism $H^1(X, \mathbb{Z}_2) \to H^1(X', \mathbb{Z}_2)$, this defines a one-to-one correspondence between spin structures on X and spin structures on X'.

(3) A theorem of Sampson and Washnitzer [29] states: If X' is obtained from X by blowing up a subvariety Y, then $H^p(X', f^*\mathscr{L}) = H^p(X, \mathscr{L})$ for any coherent sheaf $\mathscr{L}$ on X. Theorem 2.5 will then follow if we can prove that

$$H^p(X', \mathcal{O}(f^*\tfrac{1}{2}K_X)) \simeq H^p(X', \mathcal{O}(\tfrac{1}{2}K_{X'})),$$

or equivalently,

$$H^p\left(X', \mathcal{O}\left(\tfrac{1}{2}K - \frac{(m-1)}{2} E\right)\right) \cong H^p(X', \mathcal{O}(\tfrac{1}{2}K)),$$

where we write K for $K_{X'}$.

LEMMA. $H^p(X', \mathcal{O}(\frac{1}{2}K - \ell E)) \cong H^p(X', \mathcal{O}(\frac{1}{2}K - (\ell - 1)E))$ *for* $1 \leqslant \ell \leqslant (m-1)/2$.

Proof. Consider the following exact sequence of sheaves:

$$0 \to \mathcal{O}(W) \to \mathcal{O}(W \otimes \{S\}) \to \mathcal{O}(W \otimes \{S\})\,|_S \to 0,$$

where S is a nonsingular subvariety of codimension 1, $\{S\}$ is the corresponding line bundle, and W is a holomorphic vector bundle.

Put $W = \frac{1}{2}K - \ell E$, $S = Y'$, $\{S\} = E$, then we have the corresponding exact cohomology sequence:

$$\to H^p(X', \mathcal{O}(\tfrac{1}{2}K - \ell E)) \to H^p(X', \mathcal{O}(\tfrac{1}{2}K - (\ell - 1)E))$$
$$\to H^p(Y', \mathcal{O}(\tfrac{1}{2}K - (\ell - 1)E)\,|_{Y'}) \to.$$

The lemma will follow if we can prove that

$$H^p(Y', \mathcal{O}(\tfrac{1}{2}K - (\ell - 1)E)\,|_{Y'}) = 0.$$

Now $K\,|_{Y'} = K_{Y'} - E\,|_{Y'}$ and $E\,|_{Y'} = -H$, where H is the Hopf bundle over $Y' \cong \mathbb{P}(N)$. We have a holomorphic fibre bundle $\mathbb{P}^{m-1} \to Y' \to^p Y$ and so $K_{Y'} = p^*K_Y - mH - p^*\lambda^m N$. Therefore,

$$\tfrac{1}{2}K - (\ell - 1)E\,|_{Y'} \cong \tfrac{1}{2}p^*(K_Y - \lambda^m N) - \left(\frac{m-1}{2} - (\ell - 1)\right) H.$$

Let $k = (m-1)/2 - (\ell - 1)$, then $1 \leqslant k \leqslant (m-1)/2$. Consider $H^p(\mathbb{P}(N), \mathcal{O}(p^*L - kH))$ for an arbitrary holomorphic line bundle L on Y. Now $H^p(\mathbb{P}^{m-1}, \mathcal{O}(-kH)) = 0$ for $p \neq m-1$ by Kodaira's vanishing theorem and

$$H^{m-1}(\mathbb{P}^{m-1}, \mathcal{O}(-kH)) \cong H^0(\mathbb{P}^{m-1}, \mathcal{O}((k-m)H))$$

by Serre duality. Since $k \leqslant (m-1)/2$, $k - m < 0$, and so $H^{m-1}(\mathbb{P}^{m-1}, \mathcal{O}(-kH)) = 0$.

Hence $H^p(\mathbb{P}^{m-1}, \mathcal{O}(-kH)) = 0$ for all p and the E^2 term in the spectral sequence for the fibration $\mathbb{P}^{m-1} \to \mathbb{P}(N) \to Y$ and the sheaf $\mathcal{O}(p^*L - kH)$ vanishes and so

$$H^p(\mathbb{P}(N), \mathcal{O}(p^*L - kH)) = 0 \qquad \text{for all} \quad p.$$

Consequently, $H^p(Y', \mathcal{O}(\frac{1}{2}K - (\ell - 1)E)|_{Y'}) = 0$ and the lemma follows, taking $L = \frac{1}{2}(K_Y - \lambda^m N)$.

By induction on the lemma we get

$$H^p\left(X', \mathcal{O}\left(\tfrac{1}{2}K - \frac{(m-1)}{2} E\right)\right) \cong H^p(X', \mathcal{O}(\tfrac{1}{2}K)),$$

and the theorem follows.

Remark. In real dimension 2, we saw that the dimension of the space of harmonic spinors does in general depend upon the metric but is bounded above by the topological invariant $(g + 1)$; furthermore, there was no unique spin structure. In Sections 3.1–3.3 we shall see that in 3 dimensions, boundedness no longer holds.

3.1. Harmonic Spinors on S^3

The standard metric on S^3 has positive scalar curvature and so by Lichnerowicz's theorem there are no harmonic spinors. This is, however, a very special metric and if we regard S^3 as a compact Lie group ($SU(2)$, $Sp(1)$, or Spin (3)), it corresponds to the *bi*-invariant metric. We consider now only *left*-invariant metrics.

If X, Y are left-invariant vector fields and g is a left-invariant metric, then $g_p(X_p, Y_p) = g_e(X_e, Y_e)$, and since the left-invariant vector fields span the tangent space at every point p, a left-invariant metric is defined by a metric on the tangent space at the identity, i.e., the Lie algebra.

The tangent bundle is parallelized by a basis for the Lie algebra and so the spinor bundle is parallelized by the corresponding spinor basis. Hence, relative to a left-invariant metric, the Dirac operator will be a 2×2 matrix of first-order linear left-invariant differential operators, i.e., elements of the Lie algebra and constants.

Proposition 3.1. *Let g be a left-invariant metric which is diagonal with eigenvalues λ_1, λ_2, λ_3 relative to a basis $\{e_1, e_2, e_3\}$ of the Lie algebra which is orthonormal with respect to the bi-invariant metric. Then, relative* *to the corresponding spinor basis, the Dirac operator may be written*:

$$P = \begin{pmatrix} -ie_1/\sqrt{\lambda_1} & -ie_2/\sqrt{\lambda_2} + e_3/\sqrt{\lambda_3} \\ -ie_2/\sqrt{\lambda_2} - e_3/\sqrt{\lambda_3} & ie_1/\sqrt{\lambda_1} \end{pmatrix} + \frac{(\lambda_1 + \lambda_2 + \lambda_3)}{2\sqrt{\lambda_1\lambda_2\lambda_3}}.$$

Proof. We first compute the riemannian connection relative to this metric. This is defined in general by the following formula:

$$2g(X, \nabla_Z Y) = Z \cdot g(X, Y) + g(Z, [X, Y]) + Y \cdot g(X, Z)$$
$$+ g(Y, [X, Z]) - X \cdot g(Y, Z) - g(X, [Y, Z])$$

for vector fields X, Y, and Z.

For left-invariant vector fields and a left-invariant metric, this becomes:

$$2g(X, \nabla_Z Y) = g(Z, [X, Y]) + g(Y, [X, Z]) - g(X, [Y, Z])$$

since $g(X, Y)$ is constant.

Since $Ad\colon S^3 \to SO(3)$ is surjective, the basis $\{e_1, e_2, e_3\}$ satisfies the usual relations $[e_1, e_2] = 2e_3$, $[e_2, e_3] = 2e_1$, $[e_3, e_1] = 2e_2$, and hence on this basis, the riemannian connection can be computed via the above formula as:

$$\nabla_{e_1} e_1 = 0$$
$$\nabla_{e_1} e_2 = (-\lambda_1 + \lambda_2 + \lambda_3)\, e_3/\lambda_3$$
$$\nabla_{e_1} e_3 = -(-\lambda_1 + \lambda_2 + \lambda_3)\, e_2/\lambda_2$$

etc.

If $E_i = e_i/\sqrt{\lambda_i}$, then $\{E_1, E_2, E_3\}$ is an orthonormal basis relative to g and then

$$\nabla_{E_1} E_1 = 0$$
$$\nabla_{E_1} E_2 = (-\lambda_1 + \lambda_2 + \lambda_3)\, E_3/\sqrt{\lambda_1\lambda_2\lambda_3} \qquad (1)$$
$$\nabla_{E_1} E_3 = -(-\lambda_1 + \lambda_2 + \lambda_3)\, E_2/\sqrt{\lambda_1\lambda_2\lambda_3}.$$

To the basis $\{E_i\}$ of the tangent bundle, there corresponds under the spin representation a basis $\{\psi_\alpha\}$ of the spinor bundle V^+. If ω_{ij} is the connection matrix relative to the basis $\{E_i\}$, then the induced connection on V^+ is given by

$$D\psi_\alpha = \tfrac{1}{4} \sum \omega_{ij} E_i E_j \psi_\alpha$$

(see 1.1). Hence in our case,

$$\nabla_{E_1}\psi_\alpha = \frac{1}{2}\cdot\frac{(-\lambda_1+\lambda_2+\lambda_3)}{\sqrt{\lambda_1\lambda_2\lambda_3}}E_2E_3\psi_\alpha\,. \tag{2}$$

We have the Dirac operator $P\colon \Gamma(V^+)\to\Gamma(V^+)$ defined by $P\psi = \omega\sum E_i\nabla_{E_i}\psi$, where $\omega = E_1E_2E_3$ is the section of the Clifford bundle defined by the volume form, and so from (1), the action of P on a *basis* spinor ψ_α is

$$P\psi_\alpha = \omega^2\frac{(\lambda_1+\lambda_2+\lambda_3)}{2\sqrt{\lambda_1\lambda_2\lambda_3}}\psi_\alpha = -\frac{(\lambda_1+\lambda_2+\lambda_3)}{2\sqrt{\lambda_1\lambda_2\lambda_3}}\psi_\alpha\,. \tag{3}$$

Now if $\psi\in\Gamma(V^+)$, $a\in C^\infty(X)$, we have

$$P(a\psi) = \omega\sum(E_i\cdot a)\,E_i\psi + aP\psi, \tag{4}$$

where $E_i\cdot a = \langle da, E_i\rangle$, i.e., E_i acts on a as a first-order differential operator and acts on ψ by Clifford multiplication.

We take explicitly the spin representation given by

$$\omega E_1 = -i\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix};\qquad \omega E_2 = -i\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix};$$

$$\omega E_3 = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix};$$

and then from (3) and (4) we compute the action of P on the spinor $a_1\psi_1 + a_2\psi_2 = \binom{a_1}{a_2}$:

$$P\begin{pmatrix}a_1\\ a_2\end{pmatrix} = \left\{\begin{pmatrix}-iE_1 & -iE_2+E_3\\ -iE_2-E_3 & iE_1\end{pmatrix} + \frac{(\lambda_1+\lambda_2+\lambda_3)}{2\sqrt{\lambda_1\lambda_2\lambda_3}}\right\}\begin{pmatrix}a_1\\ a_2\end{pmatrix},$$

and Proposition (3.1) follows.

We restrict ourselves now to considering metrics which are left-invariant under S^3 and right-invariant under $S^1\subset S^3$. This is equivalent (up to a constant multiple) to the case $\lambda_2=\lambda_3=1$. Put $\lambda_1=\lambda^2$, then the Dirac operator becomes:

$$P = \begin{pmatrix}-ie_1/\lambda & -ie_2+e_3\\ -ie_2-e_3 & ie_1/\lambda\end{pmatrix} + (\lambda^2+2)/2\lambda.$$

Put $X = e_1$, $Z^+ = e_2 + ie_3$, $Z^- = e_2 - ie_3$, then

$$P = -i\begin{pmatrix}X/\lambda & Z^+\\ Z^- & -X/\lambda\end{pmatrix} + (\lambda^2+2)/2\lambda.$$

PROPOSITION 3.2. *The eigenvalues of P are:*

$$p/\lambda + \lambda/2 \qquad \textit{multiplicity} \quad 2p$$

$$\lambda/2 \pm \sqrt{4pq\lambda^2 + (p-q)^2}/\lambda \qquad \textit{multiplicity} \quad p+q$$

for $p, q > 0$.

Proof. Let Δ be the laplacian on functions relative to the bi-invariant metric, and let Δ act on the spinors by $\Delta \cdot \binom{a_1}{a_2} = \binom{\Delta a_1}{\Delta a_2}$. Then Δ commutes with P and we may consider P restricted to the eigenspaces of Δ. The eigenspaces of Δ acting on functions are given by the irreducible representation spaces $E \otimes E$ of $S^3 \times S^3$, where E is an irreducible representation space of S^3. There is one irreducible representation of S^3 in each dimension and these are given by the symmetric products of the two-dimensional complex representation $S^3 \cong^{\sigma} SU(2)$.

On the Lie algebra, this representation is defined by:

$$e_1 = i\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \qquad e_2 = i\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \qquad e_3 = \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}.$$

Then

$$X = i\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \qquad Z^+ = \begin{pmatrix}0 & 0\\ 2i & 0\end{pmatrix}, \qquad Z^- = \begin{pmatrix}0 & 2i\\ 0 & 0\end{pmatrix}.$$

The space of the kth symmetric power of the representation σ is spanned by monomials $x^m y^n$, where $k = m + n$, and the action of the Lie algebra on this space is given by

$$X \cdot (x^m y^n) = i(m-n)\, x^m y^n$$

$$Z^- \cdot (x^m y^n) = 2inx^{m+1} y^{n-1}$$

$$Z^+ \cdot (x^m y^n) = 2imx^{m-1} y^{n+1};$$

where $x = \binom{1}{0}$, $y = \binom{0}{1}$.

Consider now the operator

$$Q = \begin{pmatrix}\lambda^{-1}X & Z^+\\ Z^- & -\lambda^{-1}X\end{pmatrix};$$

then

$$Q^2 = \begin{pmatrix}\lambda^{-2}X^2 + Z^+Z^- & \lambda^{-1}(XZ^+ - Z^+X)\\ \lambda^{-1}(Z^-X - XZ^-) & Z^-Z^+ + \lambda^{-2}X^2\end{pmatrix}.$$

But from the commutation properties of the Lie algebra,

$$[X, Z^+] = -2iZ^+, \qquad [X, Z^-] = 2iZ^-, \qquad [Z^+, Z^-] = -4iX.$$

Hence,

$$\underset{\sim}{Q}^2 = \begin{pmatrix} \lambda^{-2}X^2 + Z^+Z^- & -2i\lambda^{-1}Z^+ \\ -2i\lambda^{-1}Z^- & \lambda^{-2}X^2 + Z^-Z^+ \end{pmatrix}$$

and

$$\underset{\sim}{Q}^2 + 2i\lambda^{-1}\underset{\sim}{Q} = \begin{pmatrix} \lambda^{-2}X^2 + \lambda^{-2}2iX + Z^+Z^- & 0 \\ 0 & \lambda^{-2}X^2 - \lambda^{-2}2iX + Z^-Z^+ \end{pmatrix}.$$

Now

$$\begin{aligned} &Z^-Z^+ \text{ acts as } -4m(n+1) \text{ on } x^my^n, \\ &Z^+Z^- \text{ acts as } -4n(m+1) \text{ on } x^my^n, \\ &X \quad\ \text{ acts as } i(m-n) \quad\ \text{ on } x^my^n. \end{aligned}$$

Thus, relative to the basis $\{x^my^n\}$, $\underset{\sim}{Q}^2 + 2i\lambda^{-1}\underset{\sim}{Q}$ is diagonal, and the eigenvalues of $\underset{\sim}{Q}$ must satisfy the equation

$$z^2 + 2i\lambda^{-1}z = -\lambda^{-2}(m-n)^2 + \lambda^{-2}(-2(m-n)) - 4n(m+1)$$

or

$$= -\lambda^{-2}(m-n)^2 + \lambda^{-2}(-2(n-m)) - 4m(n+1),$$

i.e., $z = -i\lambda^{-1} \pm i\lambda^{-1}\sqrt{(m+1-n)^2 + 4(m+1)n\lambda^2}$. Now $\psi = \binom{x^my^n}{0}$ is an eigenvector of $\underset{\sim}{Q}$ iff $Z^-(x^my^n) = 0$, i.e., iff $n = 0$ and then the eigenvalue is $\lambda^{-1}im$. Otherwise, the space generated by ψ is two dimensional and both solutions of the above equation are eigenvalues of $\underset{\sim}{Q}$. Hence the eigenvalues of P are:

$$\left.\begin{matrix} (k+1)/\lambda + \lambda/2 \\ \lambda/2 \pm \sqrt{4(m+1)\,n\lambda^2 + (m-n+1)^2}/\lambda \end{matrix}\right\} \begin{matrix} k = m+n, \\ m \geqslant 0, \quad n > 0, \end{matrix}$$

i.e., putting $p = m + 1$, $q = n$,

$$\left.\begin{matrix} p/\lambda + \lambda/2 \\ \lambda/2 \pm \sqrt{4pq\lambda^2 + (p-q)^2}/\lambda \end{matrix}\right\} p, q > 0.$$

Both $\binom{x^k}{0}$ and $\binom{0}{y^k}$ are eigenvectors corresponding to the eigenvalue $(k+1)/\lambda + \lambda/2$. Furthermore, the representation space occurs with multiplicity the dimension of the representation E $(=k+1)$ in the eigenspace of Δ, since it is also a representation space on the right. Hence the multiplicity of the eigenvalue $p/\lambda + \lambda/2$ is $2p$ and of the others $(p+q)$, which proves the proposition.

We are interested in the null space of P, i.e., the space of harmonic spinors. Thus there are harmonic spinors for the metric with eigenvalues $(\lambda^2, 1, 1)$ iff there are positive integer solutions (p, q) to the equation:

$$\lambda^2 = 2\sqrt{4pq\lambda^2 + (p-q)^2}. \tag{3.3}$$

In particular, if $\lambda = 4m$, m a positive integer, then $p = q = m$ is a solution. Thus we have the following corollary to Proposition (3.2):

COROLLARY. *Let $m > 0$ be an integer and take the metric* $\begin{pmatrix} 16m^2 & & \\ & 1 & \\ & & 1 \end{pmatrix}$ *on S^3 relative to the standard basis $\{i, j, k\}$ of the Lie algebra. Then the dimension of the space of harmonic spinors relative to this metric is $\geqslant 2m$. In particular, we can choose the metric to make* dim H *as large as we please.*

Remarks. (1) To compute the exact dimension of the space of harmonic spinors involves finding all positive integer solutions to (3.3). I owe the following observations to S. Chowla:

(a) If m is a prime $\equiv 3 \pmod 4$ and $\lambda = 4m$, then the only solution to (3.3) is $p = q = m$.

Proof. (3.3) is equivalent to

$$(p-q)^2 + 64m^2pq = 64m^4.$$

Now since $(p-q)^2 \geqslant 0$, $pq \leqslant m^2$. Furthermore, $8m \mid p - q$, so put $p - q = 8mt$, then

$$t^2 + pq = m^2,$$

and substituting for q,

$$t^2 + p^2 \equiv 0 \pmod m.$$

If $m \equiv 3 \pmod 4$, then $m \mid p$, but then $m \mid q$ and $pq \leqslant m^2$ implies $p = q = m$.

(b) If $m = 65$, then $p = q = 65$ and $p = 528$, $q = 8$ are two solutions.

(2) The solution to (3.3) given by $p = q = m$ does not arise fortuitously—it exists for geometric rather than number theoretic reasons.

The space of harmonic spinors corresponding to $p = q = m$ consists of $2m$ copies of $\begin{pmatrix} x^{m-1}y^m \\ -x^m y^{m-1} \end{pmatrix}$. Suppose that $\psi = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ is a harmonic spinor, then $X \cdot a_1 = i(m - 1 - m)a_1 = -ia_1$, $X \cdot a_2 = i(m - m + 1)a_2 = +ia_2$. Hence $\exp tX \cdot a_1 = e^{-it}a_1$, so a_1 is the pullback of a section of the homogeneous line bundle over $S^2 = S^3/S^1$ defined by the character e^{-it}, that is, H^{-1} where H is the Hopf bundle over $S^2 \cong \mathbb{P}^1$. Similarly, a_2 defines a section of H, so ψ is the pullback of a section of $H \oplus H^{-1}$—the spinor bundle on S^2 (see Sections 2.1–2.5).

Now consider Z^+ acting on the pullback a of a section of H:

$$XZ^+a = Z^+Xa - 2iZ^+a = -iZ^+a.$$

Hence $\exp tX \cdot Z^+a = e^{-it}Z^+a$ and Z^+ defines a differential operator $Z^+\colon \Gamma(H) \to \Gamma(H^{-1})$, and similarly we have $Z^-\colon \Gamma(H^{-1}) \to \Gamma(H)$. Thus $P_2 = \begin{pmatrix} & Z^+ \\ Z^- & \end{pmatrix}$ defines a differential operator on the spinor bundle of S^2—in fact a multiple of the Dirac operator.

ψ is an eigenvector of P_2 and we have found a harmonic spinor by "separation of variables"—expressed the Dirac operator $P = P_1 + P_2$, where P_1 defines an operator on S^1 and P_2 defines the Dirac operator on S^2. ψ is an eigenvector of P_2 and an eigenvector of P_1 with opposite eigenvalue. The procedure is similar to the classical construction of solutions to Laplace's equation in $\mathbb{R}^3$ (with the flat metric) by separation of variables from eigenvectors of the laplacian on S^2 and a radial differential operator.

(3) The results of Proposition 3.2 may be used to provide an example of the theorem of Atiyah, Patodi, and Singer [11] applied to the Dirac operator P on a $4n - 1$ manifold X.

We take the eigenvalues λ of P and define the difference of two zeta functions

$$\eta(s) = \sum_{\lambda \neq 0} (\operatorname{sign} \lambda) \, |\lambda|^{-s};$$

then $\eta(s)$ is finite at $s = 0$. On the other hand, we make X bound a spin manifold Y, extend the product metric near the boundary to Y, take the Pontrjagin forms on Y relative to this metric, and integrate the $\hat{A}$ polynomial in these forms over Y.

The theorem then says that

$$\eta(0) = 2\int_Y \hat{A}(p(Y)) \bmod \mathbb{Z}$$
$$= 2\Phi(X) \bmod \mathbb{Q},$$

where Φ is the Chern–Simons invariant corresponding to the $\hat{A}$ polynomial [18].

In our example, for $\lambda^2 < 16$, the positive eigenvalues are $p/\lambda + \lambda/2$ and $\lambda/2 + \sqrt{4pq\lambda^2 + (p-q)^2}/\lambda$ and the negative ones $\lambda/2 - \sqrt{4pq\lambda^2 + (p-q)^2}/\lambda$ with the appropriate multiplicities from Proposition 3.2. Hence

$$\eta(s) = \sum_{p>0} 2p(p + \lambda^2/2)^{-s}$$
$$+ \sum_{p,q>0} (p+q)[(\lambda^2/2 + \sqrt{4pq\lambda^2 + (p-q)^2})^{-s}$$
$$- (-\lambda^2/2 + \sqrt{4pq\lambda^2 + (p-q)^2})^{-s}].$$

The first term causes no problem and at $s = 0$ has the value $(\lambda^4 - 1)/6$. The second term we expand as follows. Putting

$$f(s) = \sum_{p,q>0} (p+q)(4pq\lambda^2 + (p-q)^2)^{-s},$$

we get

$$-2s\frac{\lambda^2}{2} f\left(\frac{s+1}{2}\right) - \frac{2s(s+1)(s+2)}{3!}\left(\frac{\lambda^2}{2}\right)^3 f\left(\frac{s+3}{2}\right) + g(s),$$

where for λ sufficiently small, g is analytic at $s = 0$ and $g(0) = 0$, since $f(s)$ converges absolutely for $\operatorname{Re} s > 3/2$. Computing the residues of f at $s = 1/2$ and $s = 3/2$, we finally obtain the following expression for $\eta(0)$:

$$\eta(0) = (-1 + 2\lambda^2 - \lambda^4)/6.$$

Modulo $\mathbb{Q}$ (and a sign convention), this agrees with the Chern–Simons invariant for this family of metrics as computed in [18], taking account of the fact that $\hat{A}(p_1) = -p_1/24$.

3.2. Hopf Surfaces

Let $X = S^1 \times S^3$. Let e_0 be an invariant vector field on S^1 and e_1, e_2, e_3 the standard orthonormal vector fields on S^3. Then

$$J(e_0) = -e_1 \qquad J(e_2) = -e_3$$
$$J(e_1) = e_0 \qquad J(e_3) = e_2$$

defines an almost complex structure on X which is integrable and gives a complex structure. X is a Hopf manifold.

The left-invariant riemannian metric defined by the matrix

$$\left(\begin{array}{c|c|cc} \lambda & & & \\ \hline & \lambda & & \\ \hline & & 1 & \\ & & & 1 \end{array}\right)$$

(with respect to the basis $\{e_0, e_1, e_2, e_3\}$) is then *hermitian*. But this is the product of an invariant metric on S^1 and the metric we were considering in Proposition 3.2. Hence by the product formula for harmonic spinors (1.1, Remark 4), we can make the dimension of the space of harmonic spinors on X as large as we please by choosing λ suitably.

On the other hand, $p_g = 0$ for X and so if $h^p = \dim H^p(X, \mathcal{O}(\frac{1}{2}K))$, $h^0 = 0$. Furthermore, by Serre duality and the Riemann–Roch theorem for complex manifolds, $2h^0 - h^1 = \hat{A}(X) = 0$ since sign $X = 0$, and so $h^p = 0$ for all p.

We see here the necessity of the Kähler condition in Theorem 2.2. In fact, X is the simplest example of a non-Kähler compact complex manifold (since $H^2(X, \mathbb{Z}) = 0$, X cannot be Kähler).

3.3. Scalar Curvatures of S^3

The scalar curvature R of a left-invariant metric is a constant and so since we have harmonic spinors relative to metrics within the family of Proposition 3.2, these must have nonpositive scalar curvature by the theorem of Lichnerowicz. It is a matter of interest then to compute the scalar curvature of a left-invariant metric on S^3.

The curvature tensor is given by:

$$R(X, Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z.$$

Hence from (i) in Proposition 3.1,

$$\begin{aligned} &R(E_1, E_2)\, E_2 \\ &= \nabla_{E_1}\nabla_{E_2}E_2 - \nabla_{E_2}\nabla_{E_1}E_2 - \nabla_{[E_1,E_2]}E_2 \\ &= -\nabla_{E_2}\frac{(-\lambda_1+\lambda_2+\lambda_3)}{\sqrt{\lambda_1\lambda_2\lambda_3}}\,E_3 - \frac{2\lambda_3}{\sqrt{\lambda_1\lambda_2\lambda_3}}\nabla_{E_3}E_2 \\ &= \frac{1}{\lambda_1\lambda_2\lambda_3}\{-(-\lambda_1+\lambda_2+\lambda_3)(\lambda_1-\lambda_2+\lambda_3) + 2\lambda_3(\lambda_1+\lambda_2-\lambda_3)\}\,E_1\,. \end{aligned}$$

Now the scalar curvature $R = \sum_{i,j} (R(E_i, E_j)E_j, E_i)$. But from the symmetry

$$(R(X, Y)Z, W) = (R(Z, W)X, Y)$$

and $R(X, Y) = -R(Y, X)$, we get

$$R = 2\{(R(E_1, E_2)\,E_2, E_1) + (R(E_2, E_3)\,E_3, E_2) + (R(E_3, E_1)\,E_1, E_3)\}.$$

But from the above formula,

$$(R(E_1, E_2)\,E_2, E_1) = \{-(\sigma_1 - 2\lambda_1)(\sigma_1 - 2\lambda_2) + 2\lambda_3(\sigma_1 - 2\lambda_3)\}/\sigma_3\,,$$

where σ_i is the ith elementary symmetric function in $\{\lambda_1, \lambda_2, \lambda_3\}$, and so

$$\begin{aligned} R &= 2\{-3\sigma_1^2 - 4\sigma_2 + 4\sigma_1^2 + 2\sigma_1^2 - 4(\sigma_1^2 - 2\sigma_2)\}/\sigma_3 \\ &= 2(4\sigma_2 - \sigma_1^2)/\sigma_3\,. \end{aligned} \tag{3.4}$$

The restricted family of Proposition 3.2 was given by $\lambda_1 = \lambda^2$, $\lambda_2 = \lambda_3 = 1$. Hence in this case,

$$\begin{aligned} R &= 2(4(2\lambda^2 + 1) - (\lambda^2 + 2)^2)/\lambda^2 \\ &= 2(4 - \lambda^2) \end{aligned} \tag{3.5}$$

Remarks. (1) Consider again the equation (3.3), i.e.,

$$4(p - q)^2 + 16pq\lambda^2 = \lambda^4.$$

Then $16pq \leqslant \lambda^2$, so there are no positive integer solutions (p, q) (and hence no harmonic spinors) if $\lambda^2 < 16$. From (3.5), we see that if $\lambda^2 < 4$, $R > 0$, and so this result is compatible with Lichnerowicz's theorem. Also, if $\lambda^2 = 4$, $R = 0$, and then the result is compatible with Theorem 1.2, since $S^3 \times S^1$ is not a Kähler manifold. In fact there are no harmonic spinors until $\lambda^2 = 16$ and $R = -24$, and then there are two linearly independent harmonic spinors since $p = q = 1$ is the unique solution to (3.3).

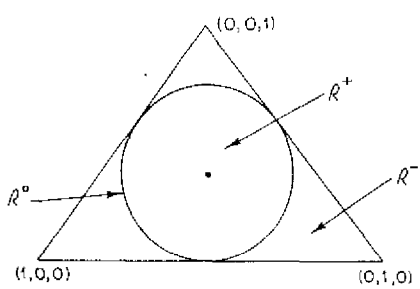
(2) We may regard the space of left-invariant metrics on S^3 (up to isometry by conjugation) as parametrized by the eigenvalues $(\lambda_1, \lambda_2, \lambda_3)$, where $\lambda_i > 0$. The space of such metrics of positive scalar curvature ($\mathscr{R}^+$) is then given from (3.4) by:

$$\{(\lambda_1, \lambda_2, \lambda_3) \mid 4(\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1) > (\lambda_1 + \lambda_2 + \lambda_3)^2\},$$

i.e.,

$$\{(\lambda_1, \lambda_2, \lambda_3) \mid \lambda_1^2 + \lambda_2^2 + \lambda_3^2 < \tfrac{1}{2}(\lambda_1 + \lambda_2 + \lambda_3)^2\}.$$

This is the interior of a circular cone with axis (1, 1, 1). Up to multiplication of the metric by a constant, we may represent the left-invariant metrics by barycentric coordinates as the interior of a 2-simplex. The space of metrics of positive scalar curvature is then given by the interior of the inscribed circle:



Note that $\mathscr{R}^+$ is contractible.

4.1. Skew-Adjoint Fredholm Operators

Let H be a real infinite-dimensional Hilbert space which is a module for the real Clifford algebra C_{k-1} such that $J_i^* = -J_i$, where $\{J_1, \ldots, J_{k-1}\}$ is an orthonormal basis for $\mathbb{R}^{k-1}$. Let $\mathscr{F}$ denote the space of skew-adjoint Fredholm operators on H and $\mathscr{F}^k$ the subset of all $A \in \mathscr{F}$ such that $AJ_i = -J_iA$ $(1 \leqslant i \leqslant k-1)$ (see Atiyah and Singer [13] for details).

If $k \equiv -1 \pmod 4$ and $A \in \mathscr{F}^k$, define $w(A) = J_1J_2, \ldots, J_{k-1}A$. Then $w(\mathscr{F}^k)$ is the space of *self-adjoint* Fredholm operators commuting with C_{k-1}. In fact, since $C_6 \cong \operatorname{End}(\mathbb{R}^8)$ and $C_3 \cong \mathbb{H}$, we can identify $w(\mathscr{F}^{8k-1})$ with the space of all real self-adjoint Fredholm operators and $w(\mathscr{F}^{8k-5})$ with all quaternionic self-adjoint Fredholm operators. $\mathscr{F}^k$ is the union of components $\mathscr{F}_+{}^k$, $\mathscr{F}_-{}^k$, and $\mathscr{F}_*{}^k$, where $w(\mathscr{F}_+{}^k)$ (resp. $w(\mathscr{F}_-{}^k)$) is the space of all essentially positive (resp. negative) self-adjoint operators.

If $k \not\equiv -1 \pmod 4$, then set $\mathscr{F}_*{}^k = \mathscr{F}^k$. It is shown in [13] that $\mathscr{F}_*{}^k$ is a classifying space for KR^{-k}. Hence, given any compact space X and a continuous map $A\colon X \to \mathscr{F}_*{}^k$, we have a well-defined homotopy invariant

$$\operatorname{index} A \in KR^{-k}(X).$$

PROPOSITION 4.1. (1) *If $A(x)$ is invertible for all $x \in X$, then index $A = 0$.*

(2) *If $k \equiv -1 \pmod 4$ and the rank of $A(x)$ is constant, then index $A = 0$.*

Proof. (1) This is proved in [13] and follows from Kuiper's theorem.

(2) If rank $A(x)$ is constant, we can define a continuous map

$$X \to \mathscr{K}$$
$$x \to P(x),$$

where $\mathscr{K}$ is the space of compact operators and $P(x)$ is the orthogonal projection operator onto the kernel of $w(A(x))$. $P(x)$ is selfadjoint and commutes with C_{k-1} since $H = \ker A(x) \oplus (\ker A(x))^\perp$ is a decomposition of C_{k-1}-modules. Consider now

$$B(x, t) = w(A(x)) + tP(x).$$

$B(x, t)$ is self-adjoint, Fredholm, and commutes with C_{k-1}. $B(x, 0) = w(A(x))$, and $B(x, 1)$ is invertible. Hence A retracts to a map into the invertible elements, which is homotopic to zero by part 1. Hence index $A = 0$.

We may equivalently regard $\mathscr{F}^k$ in the following way (see [13]): Let $H = H^0 \oplus H^1$ be a $\mathbb{Z}_2$-graded C_k-module. Consider the set of skew-adjoint Fredholm operators A such that $A\colon H^0 \to H^1$ and $H^1 \to H^0$

and $AJ_i = -J_iA$ ($1 \leqslant i \leqslant k$). Then $A \mapsto J_kA|_{H^0}$ gives an isomorphism of the above set with $\mathscr{F}^k(H^0)$ (H^0 is a $C_k^0 \simeq C_{k-1}$ module). With this description we can define index A as the index of a family of operators parametrized by $X \times \mathbb{R}^k$: given A: $X \to \mathscr{F}_*^k$, we define a map

$$B: X \times \mathbb{R}^k \to \mathscr{F}^k$$
$$(x, t) \mapsto A(x) + C(t),$$

where $C(t)$ denotes Clifford multiplication by $t \in \mathbb{R}^k$ and we have identified $\mathscr{F}_*^k$ with the above set. Since $C(t)^{-1} \cdot A(x)$ is skew-adjoint, $B(x, t)$ is invertible for $t \neq 0$ and hence defines an element in $KR(X \times \mathbb{R}^k) \simeq KR^{-k}(X)$ which is the index defined above.

4.2. Families of Dirac Operators

The prototype for the sort of operator described in **4.1** is given by the real Dirac operator on a spin manifold, which we define as follows.

Let X be a spin manifold of dimension k, define the real spinor bundle by $V = \tilde{E} \times_{\text{Spin}} C_k$, where $\tilde{E}$ is the principal spin bundle and $\text{Spin}(k) \subset C_k$ acts on C_k by left multiplication. V decomposes into $V^0 \oplus V^1$ corresponding to the even and odd parts of C_k. We can multiply sections of V on the *left* by sections of $C(T)$, the Clifford algebra bundle of the tangent bundle, and on the *right* by elements of C_k. The two multiplications commute.

We have a Dirac operator $P: \Gamma(V) \to \Gamma(V)$ defined in the usual way, with $P: \Gamma(V^0) \to \Gamma(V^1)$ and $\Gamma(V^1) \to \Gamma(V^0)$. The complexification of P is just a certain number of copies of the Dirac operator defined in Sections 1.1–1.4, which is associated to a complex *irreducible* representation of $\text{Spin}(k)$.

P is not a bounded operator on the space of sections of V, but if we set $\underset{\sim}{Q} = (1 + D^*D)^{-1/4}$, where D: $\Gamma(V) \to \Gamma(V \otimes T^*)$ is the covariant derivative, and then put $P_0 = \underset{\sim}{Q}P\underset{\sim}{Q}^*$, we get a bounded, zero-order operator with isomorphic kernel and the same symbol (restricted to the unit sphere bundle).

P_0 is self-adjoint and commutes with Clifford multiplication on the right. Hence we define

$$\left.\begin{aligned} P_1\psi^0 &= P_0\psi^0 \\ P_1\psi^1 &= -P_0\psi^1 \end{aligned}\right\} \psi^i \in \Gamma(V^i).$$

P_1 is now bounded, skew-adjoint, and anti-commutes with $\mathbb{R}^k \subset C_k$. Furthermore, since P is a first-order differential operator, it cannot define an essentially positive or negative operator. A family of such operators, parametrized by Y, therefore has an index in $KR^{-k}(Y)$. We compute this index via the Atiyah–Singer index theorem for families of operators [14].

Let $X \to Z \to^p Y$ be a compact fibre bundle with fibre X a spin manifold of dimension k, and tangent bundle along the fibres spin. Introduce a continuous family of metrics in the fibres and take the Dirac operator in each fibre relative to the metric g_y. Then $y \mapsto P_1(y)$ defines a family of operators $A\colon Y \to \mathscr{F}_*{}^k$.

PROPOSITION 4.2. *index* $A = p!(1)$ *where* $p! : KR(Z) \to KR^{-k}(Y)$ *is the direct image homomorphism for spin maps.*

Proof. By the index theorem, the analytical index (index A) is equal to the topological index of the symbol class of the family of complexes $P_1(y, t)\colon \Gamma(V^0) \to \Gamma(V^1)$ $((y, t \in Y \times \mathbb{R}^k)$ in $KR(T_F Z \times \mathbb{R}^k)$, where $T_F Z$ is the tangent bundle along the fibres with involution given by the antipodal map $\xi \mapsto -\xi$. We first calculate this symbol class.

The symbol of the Dirac operator P is given by $L(i\xi_z)\colon V_z{}^0 \otimes \mathbb{C} \to V_z{}^1 \otimes \mathbb{C}$ where $L(\alpha_z)$ denotes left Clifford multiplication by α_z. Thus the symbol of the zero-order operator P_1 is given on the whole spinor bundle by:

$$\sigma(\xi_z, t)\colon V_z \otimes \mathbb{C} \to V_z \otimes \mathbb{C},$$

$$\sigma(\xi_z, t) \cdot \psi^0 = L(i\xi_z) \cdot \psi^0 + R(t) \cdot \psi^0,$$

$$\sigma(\xi_z, t) \cdot \psi^1 = -L(i\xi_z) \cdot \psi^1 + R(t) \cdot \psi^1,$$

where $R(t)$ denotes right Clifford multiplication by $t \in \mathbb{R}^k$. Now

$$\sigma(\xi_z, t)^2 = -(\xi_z, \xi_z) - (t, t)$$

and

$$\bar{\sigma}(\xi_z, t) = \sigma(-\xi_z, t),$$

and so with this Clifford multiplication, $C_k \otimes \mathbb{C}$ is a graded Spin(k, k) module of dimension 2^k. The symbol class of $P_1(y, t)$ is thus the Bott class in the Thom isomorphism for the Spin(k, k) bundle $T_F Z \times \mathbb{R}^k \to Z$ (see Atiyah [6]).

Now let N denote the normal bundle along the fibres of a fibre-wise embedding:

$$\begin{array}{ccc} Z & \hookrightarrow & Y \times \mathbf{R}^{8\ell+k} \\ & {}_{p}\searrow \quad \swarrow_{p_1} & \\ & Y & \end{array}$$

and let β:

$$KR(X) \xrightarrow{\operatorname{Spin}(p,q)} KR(E)$$

denote the Thom isomorphism in KR-theory where E is a $\operatorname{Spin}(p, q)$ bundle over X and $p \equiv q \pmod 8$.

Consider the following commutative diagram:

$$\begin{array}{ccccc} & & \overset{p!}{\longrightarrow} & & \\ KR(Z) & \xrightarrow{\operatorname{Spin}(0,8\ell)} & KR(N) & \xrightarrow{i!} & KR(Y \times \mathbb{R}^{8\ell+k}) \\ \beta \downarrow \operatorname{Spin}(k,k) & & \downarrow \operatorname{Spin}(8\ell+k,k) & & \downarrow \operatorname{Spin}(8\ell+k,k) \\ KR(T_F Z \times \mathbb{R}^k) & \xrightarrow{\operatorname{Spin}(8\ell,8\ell)} & KR(TN \times \mathbb{R}^k) & \xrightarrow{j!} & KR(Y \times T\mathbb{R}^{8\ell+k} \times \mathbb{R}^k) \\ & & \underset{\text{ind}}{\longrightarrow} & & \end{array}$$

($i!$ and $j!$ are induced by open inclusions).

We see that $p!(1) = \operatorname{ind} \beta(1)$. But $\beta(1)$ is the Bott class which we have seen is the symbol class of $P_1(y, t)$. Hence index $A = p!(1)$.

Remarks. (1) Take $Y =$ pt. in Proposition 4.2, then the Dirac operator on X has an index in $KR^{-k}(\text{pt.})$, given by $f!(1)$ where $f: X \to \text{pt.}$

$$KR^{-4m}(\text{pt.}) \cong \mathbb{Z} \qquad \text{and then} \quad f!(1) = \hat{A}(X) \quad \text{or} \quad \tfrac{1}{2}\hat{A}(X)$$

$$KR^{-(8m+1)}(\text{pt.}) \cong \mathbb{Z}_2 \qquad \text{and then} \quad f!(1) = \dim H \pmod 2$$

$$KR^{-(8m+2)}(\text{pt.}) \cong \mathbb{Z}_2 \qquad \text{and then} \quad f!(1) = \dim H^+ \pmod 2$$

(see Atiyah and Singer [14]).

We define $\alpha(X) = f!(1) \in KR^{-n}(\text{pt.})$ for a spin manifold X of dimension n. Then

$$\alpha(X \times Y) = \alpha(X) \cdot \alpha(Y),$$
$$\alpha(X \# Y) = \alpha(X) + \alpha(Y),$$

where # denotes connected sum. In fact, α defines a ring homomorphism from the spin cobordism ring Ω_*^{Spin} to $KR^{-*}(\text{pt.})$.

(2) Let $X^k \to Z \to^p Y^m$ be a differentiable fibre bundle with Y and Z spin manifolds. Then spin structures on Y and Z induce a spin structure on the tangent bundle along the fibres T_F and if f: $Y \to$ pt.,

$$\alpha(Z) = (fp)!\,(1) = f!(p!(1)) \in KR^{-k-m}(\text{pt.}),$$

where the direct image homomorphisms are taken relative to the spin structures on Y and T_F. In particular, if $p!(1) = 0$, then $\alpha(Z) = 0$, and we have the following proposition:

PROPOSITION 4.3. *Let $X \to Z \to Y$ be a differentiable fibre bundle with Y, Z spin and $\alpha(Z) \neq 0$. Then for some spin structure on X,*

(1) *X admits harmonic spinors relative to some riemannian metric,*

(2) *If* $\dim X \equiv -1 \pmod 4$, *the dimension of the space of harmonic spinors depends upon the metric.*

Proof. (1) Suppose X admits no harmonic spinors relative to any metric, then the family of Dirac operators is invertible, and so by Proposition 4.1 (part 1), the index of the family is zero and hence from Proposition 4.2 and Remark 2 above, $\alpha(Z) = 0$, which contradicts the hypothesis.

(2) Similarly, if the Dirac operators have the same rank, then by part 2 of Proposition 4.1, the index is zero and so $\alpha(Z) = 0$.

In the next section we shall construct examples where $\alpha(Z) \neq 0$.

4.3. GROMOLL GROUPS AND FIBRE BUNDLES

(For the constructions below, we refer to Antonelli, Burghelea, and Kahn [3–5].)

Let Γ^n denote the Kervaire–Milnor group of exotic n-spheres. We have a surjective homomorphism

$$T\colon \pi_0(\text{Diff } S^n) \to \Gamma^{n+1}$$

defined by $T(f) = D^{n+1} \cup_f D^{n+1}$.

Novikov defined a homomorphism

$$\lambda_i\colon \pi_i(\text{Diff } S^n) \to \Gamma^{n+i+1}$$

as follows. Let $\varphi\colon D^i \to \operatorname{Diff} S^n$ represent $[\varphi] \in \pi_i(\operatorname{Diff} S^n)$, where $\varphi(S^{i-1}) = \mathrm{id}$. Then φ defines a diffeomorphism of $D^i \times S^n$ which is the identity on the boundary. We represent S^{i+n} as $S^{i-1} \times D^{n+1} \cup D^i \times S^n$ and extend the diffeomorphism trivially over $S^{i-1} \times D^{n+1}$ to obtain a diffeomorphism of S^{i+n} which via T defines an element of Γ^{n+i+1}.

The image of λ_i in Γ^{n+i+1} is the $(i+1)$th *Gromoll group* Γ^{n+i+1}_{i+1}. We have a filtration

$$0 = \Gamma^n_{n-1} \subset \cdots \subset \Gamma_k{}^n \subset \cdots \subset \Gamma_1{}^n = \Gamma^n.$$

PROPOSITION 4.4. *If* $\sum \in \Gamma^{n+i+1}_{i+1}$, *then for any manifold* X^n, $Z = X^n \times S^{i+1} \# \sum$ *fibres differentiably over* S^{i+1} *with fibre* X^n.

Proof. Let $\operatorname{Diff}(S^n, D_+{}^n) \subset \operatorname{Diff} S^n$ denote the group of orientation-preserving diffeomorphisms of S^n which leave fixed the upper hemisphere $D_+{}^n$. Then by restriction, λ_i defines a homomorphism

$$\mu_i\colon \pi_i(\operatorname{Diff}(S^n, D_+{}^n)) \to \Gamma^{n+i+1}.$$

It is shown in [5] that $\Gamma^{n+i+1}_{i+1} = im\, \mu_i$. This follows essentially from the fact that the map $SO(n+1) \times \operatorname{Diff}(S^n, D_+{}^n) \to \operatorname{Diff} S^n$ defined by group multiplication is a homotopy equivalence, but every orthogonal diffeomorphism of S^n extends to D^{n+1} and so goes to zero in Γ^{n+i+1}.

Now we have a homomorphism

$$E\colon \operatorname{Diff}(S^n, D_+{}^n) \to \operatorname{Diff} X^n$$

for any n-manifold X by letting $f \in \operatorname{Diff}(S^n, D_+{}^n)$ act on an embedded disc $D^n \subset X$. Hence we have an induced homomorphism

$$E_*\colon \pi_i(\operatorname{Diff}(S^n, D_+{}^n)) \to \pi_i(\operatorname{Diff} X).$$

An element $E_*[\varphi] \in \pi_i(\operatorname{Diff} X)$ defines a fibre bundle over S^{i+1} by $Z = X \times D^{i+1} \cup_\varphi X \times D^{i+1}$ with $\varphi\colon D^i \to \operatorname{Diff}(S^n, D_+{}^n) \to \operatorname{Diff} X$ such that $\varphi(S^{i-1}) = \mathrm{id}$. The bundle is then trivial outside $D^i \times I \subset S^{i+1}$ and so Z is obtained from $X \times S^{i+1} = X \times D^{i+1} \cup_{id} X \times D^{i+1}$ by removing a disc $D^{n+i+1} \simeq D^n \times D^i \times I$ and attaching another via the diffeomorphism of the boundary given by:

$$\begin{array}{ll} \mathrm{id} & \text{on } S^{n-1} \times D^i \times I \\ \mathrm{id} & \text{on } D^n \times S^{i-1} \times I \\ \{\mathrm{id}, \varphi\} & \text{on } D^n \times D^i \times S^0. \end{array}$$

But this is the diffeomorphism of S^{i+n} which defines the Novikov map λ_i, hence $Z = X \times S^{i+1} \# \mu_i([\varphi])$, so if $\sum \in \Gamma_{i+1}^{n+i+1}$, then $\sum = \mu_i([\varphi])$ for some φ and then $Z = X \times S^{i+1} \# \sum$.

Now $\alpha(Z) = \alpha(X) \cdot \alpha(S^{i+1}) + \alpha(\sum) = \alpha(\sum)$, since $\alpha(S^n) = 0$ by Lichnerowicz's theorem for example (if $n = 1$, we take the spin structure which bounds, i.e., that corresponding to the nontrivial lifting of the trivial principal bundle). But it is well-known that in dimensions $8k + 1$, $8k + 2$, there exist exotic spheres $\sum^n$ for which $\alpha(\sum) \neq 0$. Milnor [27] showed this for $n = 9, 10, 17$, and 18 and proved the general case would follow from the following: For $n \equiv 1 \pmod 8$, there exists a map $f: S^{8r+n} \to S^{8r}$ so that the induced map $f^*: \widetilde{KR}(S^{8r}) \to \widetilde{KR}(S^{8r+n}) \cong Z_2$ is nonzero. This was proved by Adams [1]. See also Anderson, Brown, and Peterson [2].

In fact, such spheres form a coset of the subgroup of index 2 $\Gamma^n_{\mathrm{Spin}} \subset \Gamma^n$ of spheres which bound spin manifolds.

Suppose $\Gamma^{n+i+1}_{i+1}/\Gamma^{n+i+1}_{i+1} \cap \Gamma^{n+i+1}_{\mathrm{Spin}} \neq \{0\}$; then by Proposition 4.4, if X^n is any spin manifold, we have a differentiable fibre bundle $X^n \to Z \to S^{i+1}$ with $\alpha(Z) \neq 0$ (if $n + i + 1 \equiv 1$ or $2 \pmod 8$), so to construct examples we have to know which Gromoll groups contain spheres which do not bound spin manifolds.

We know that $\Gamma_1^{n+1} = \Gamma^{n+1}$, but as pointed out in [3], we also have $\Gamma_2^{n+1} = \Gamma_1^{n+1} = \Gamma^{n+1}$, which follows from a theorem of Cerf on isotopy and pseudo-isotopy:

Recall that a pseudo-isotopy is an element of $\mathrm{Diff}(X \times I, X \times \{0\})$ and Cerf's theorem [17] states that for a simply connected manifold X, the group of pseudo-isotopies is connected. We have an exact sequence:

$$\mathrm{Diff}(X \times I, X \times \{0, 1\}) \to \mathrm{Diff}(X \times I, X \times \{0\}) \to \mathrm{Diff}\, X$$

and a corresponding exact sequence of homotopy groups:

$$\longrightarrow \pi_1(\mathrm{Diff}\, X) \xrightarrow{\alpha} \pi_0(\mathrm{Diff}(X \times I, X \times \{0, 1\})) \longrightarrow \pi_0(\mathrm{Diff}(X \times I, X \times \{0\}))$$
$$\longrightarrow \pi_0(\mathrm{Diff}(X \times I, X \times \{0\})) = \{0\}$$

by Cerf and so α is surjective.

Consider now $f \in \mathrm{Diff}(S^n, D_+{}^n)$. f defines a diffeomorphism of $S^{n-1} \times I \subset S^n$ which is the identity on the boundary and thus from the surjectivity of α is isotopic in $\mathrm{Diff}(S^{n-1} \times I, S^{n-1} \times \{0, 1\})$ to a diffeomorphism defined by $\varphi: I \to \mathrm{Diff}\, S^{n-1}$ with $\varphi(\{0, 1\}) = \mathrm{id}$. By extending the isotopy trivially outside $S^{n-1} \times I$, we see that f is isotopic as an element of Diff S^n to the extension of φ which occurs in the Novikov homomorphism, i.e., $T(f) = \lambda_1(\varphi)$, so $\Gamma_1^{n+1} = \Gamma_2^{n+1}$.

For any spin manifold X, we now have differentiable fibre bundles

$$X^n \to Z \to S^1 \qquad (n \equiv 0, 1 \pmod 8)$$

$$X^n \to Z \to S^2 \qquad (n \equiv -1, 0 \pmod 8)$$

for which $\alpha(Z) \neq 0$.

Hence from Proposition 4.3, we can state the following result:

THEOREM 4.5. (1) *Let X be any spin manifold of dimension* 0 *or* ± 1 (mod 8). *Then X admits harmonic spinors with respect to some metric.*

(2) *If* $\dim X \equiv -1 \pmod 8$, *the dimension of the space of harmonic spinors depends upon the metric.*

We have said nothing so far about introducing a family of metrics along the fibres. This can always be done (for example, taking the metric induced from one on the total space), but in the above examples we can do it in an explicit way.

Let the bundle be defined by a map

$$\varphi\colon S^i \to \mathrm{Diff}(S^n, D_+{}^n) \to \mathrm{Diff}\, X^n.$$

Now choose a fixed metric g on X and consider the following continuous family of metrics parametrized by the disc D^{i+1}:

$$g(r, u) = (1 - r)g + r\varphi(u)^*g,$$

where r is the radius and f^*g is the pulled back metric for $f \in \mathrm{Diff}\, X^n$. Since $\varphi(u)$ is the identity outside the disc $D \subset X$, the metric is unchanged outside D. If we take two copies of D^{i+1} with the family $g(r, u)$ on one and the trivial family g on the other, then identifying via $\varphi(u)$, we have introduced a continuous family of metrics in the fibres of the bundle $X \to Z \to S^{i+1}$.

We thus see that any variation of the dimension of the space of harmonic spinors detected by the above examples is caused by *altering the metric in a neighborhood of a point.*

Remarks. (1) Although we have seen that S^3 admits harmonic spinors relative to some metrics, we cannot detect this by the above method. This follows from work of Akiba, Morlet, and Rourke (see [5]) who show that $\mathrm{Diff}^0 S^3$ retracts onto $SO(4)$ and hence $\Gamma_{n-3}^3 = \{0\}$.

(2) From Theorem 4.5, we deduce that $\dim H$ varies for the standard spheres $S^n (n \equiv 0, \pm 1 \pmod 8)$ since we know there are no harmonic spinors relative to the standard metric. Using the results of Sections 3.1–3.3 on S^3 and the product formula, we can now exhibit explicitly simply connected spin manifolds in all dimensions >5 for which the dimension of the space of harmonic spinors depends upon the metric:

dim	
$8k$	S^{8k}
$8k+1$	S^{8k+1}
$8k+2$	$S^{8k-1} \times S^3$
$8k+3$	$S^{8k} \times S^3$
$8k+4$	$S^{8k+1} \times S^3$
$8k+5$	$S^{8k-1} \times S^3 \times S^3$
$8k+6$	$S^{8k} \times S^3 \times S^3$
$8k+7$	S^{8k+7}

(3) The exotic spheres Σ for which $\alpha(\Sigma) \neq 0$ are interesting in their own right: *they do not admit any metric of positive scalar curvature.* If they did, then by Lichnerowicz's theorem there would be no harmonic spinors and so $\alpha(\Sigma)$ (which is the mod 2 dimension of the space of harmonic spinors) would be zero. In [3], Antonelli, Burghelea, and Kahn raised the question: "Can every sphere in $\Gamma_k{}^n$ be δ_k-pinched?" If a manifold has positive sectional curvature, it certainly has positive scalar curvature and so these examples provide a strong negative answer.

4.4. Metrics of Positive Scalar Curvature

Let X be a compact manifold and $\mathscr{R}(X)$ the space of all riemannian metrics on X. Let $\mathscr{R}^+(X) \subset \mathscr{R}(X)$ be the subspace of all metrics with scalar curvature $R \geqslant 0$ $(R \not\equiv 0)$. Note that $\mathscr{R}^+(X)$ may be empty, for example, when X is a spin manifold with $\alpha(X) \neq 0$.

The space $\mathscr{R}$ is convex and hence contractible, but $\mathscr{R}^+$ is not necessarily trivial topologically: we have the following proposition:

PROPOSITION 4.6. *If X is a spin manifold of dimension k, there is a homomorphism (for each spin structure)*

$$A: \pi_{n-1}(\mathscr{R}^+(X)) \to KR^{-k-n}(\text{pt.})$$

Proof. The space of riemannian metrics $\mathscr{R}(X)$ is contractible and so

$$\pi_{n-1}(\mathscr{R}^+) \cong \pi_n(\mathscr{R}, \mathscr{R}^+).$$

Let $f: (D^n, S^{n-1}) \to (\mathscr{R}, \mathscr{R}^+)$ represent an element $[f] \in \pi_n(\mathscr{R}, \mathscr{R}^+)$. To each metric we associate the real Fredholm operator P_1 defined in 4.2. Thus f defines a map $\tilde{f}: D^n \to \mathscr{F}_*{}^k$. If $x \in S^{n-1}$, $f(x) \in \mathscr{R}^+$ and so by Lichnerowicz's theorem, $\tilde{f}(x) \in \mathscr{F}_*{}^k$ is invertible. $\mathscr{F}_*{}^k$ is a classifying space for KR^{-k} and the set of invertible elements in $\mathscr{F}_*{}^k$ is contractible, hence the homotopy class of $\tilde{f}$ defines an element

$$A[f] \in KR^{-k}(D^n, S^{n-1}) \cong KR^{-k-n}(\text{pt.}).$$

A is easily seen to be a homomorphism.

The homomorphism A is defined analytically, but in certain circumstances $A[f]$ may be determined topologically. Suppose $\mathscr{R}^+ \neq \varnothing$ and let us fix $g \in \mathscr{R}^+$. If $h \in \text{Diff}\, X$, the metric h^*g is also contained in $\mathscr{R}^+$. We then get a map

$$T: \text{Diff}\, X \to \mathscr{R}^+(X)$$
$$h \mapsto h^*g$$

and a homomorphism

$$B: \pi_{n-1}(\text{Diff}\, X) \xrightarrow{T_*} \pi_{n-1}(\mathscr{R}^+(X)) \to KR^{-k-n}(\text{pt.}).$$

Given $\varphi: S^{n-1} \to \text{Diff}\, X$, we have the family of metrics $\varphi(u)^*g$ on S^{n-1} which we extend to D^n, but this corresponds to introducing a family of metrics on the fibre bundle $X \to Z \to S^n$ and $B[\varphi]$ is then clearly given by the analytical index of the family. So if Z is a spin manifold, we can use the index theorem to identify $B[\varphi]$ with $\alpha(Z)$. In particular, from the examples of Theorem 4.5, we can state the following.

THEOREM 4.7. *Let X be a spin manifold such that $\mathscr{R}^+(X) \neq \varnothing$, then*

(1) $\pi_0(\mathscr{R}^+(X)) \neq 0$ *for* $\dim X = 8k, 8k+1$

(2) $\pi_1(\mathscr{R}^+(X)) \neq 0$ *for* $\dim X = 8k-1, 8k$.

Note that on S^3, the space of left-invariant metrics of positive scalar curvature is contractible (see Sections 3.1–3.3).

4.5. Blowing Up and Down

In the previous section we used the index theorem to obtain differential geometric information (scalar curvature) from differential topological data (spin cobordism invariants). We can equally well run the machine backwards and use differential geometric data to prove topological results. We shall find next invariants of "blowing up" by applying the following lemma.

Lemma (4.8). (1) *Let $E \to Y$ be a k-dimensional quaternionic vector bundle, and let $\mathbb{H}P(E \oplus 1) \to^p Y$ be the quaternionic projective bundle of $E \oplus 1$. Then the map p is KR-oriented and*

$$p!(1) = 0 \in KR^{-4k}(Y).$$

(2) *Let $E \to Y$ be a k-dimensional complex vector bundle and $\mathbb{C}P(E \oplus 1) \to^p Y$ the projective bundle of $E \oplus 1$. Then p is K-oriented and*

$$p!([H^{-1}]) = 0 \in K^{-2k}(Y) \cong K(Y),$$

where $[H] \in K(\mathbb{C}P(E \oplus 1))$ is the class of the Hopf bundle.

Proof. (1) Let E be a k-dimensional quaternionic vector bundle and H a quaternionic line bundle. We can define a real oriented $4k$-dimensional vector bundle $E \cdot H$ by the inclusion

$$\mathrm{Sp}(k) \cdot \mathrm{Sp}(1) \hookrightarrow SO(4k)$$

defined by left multiplication by an element of $\mathrm{Sp}(k)$ and right multiplication by an element of $\mathrm{Sp}(1)$. From the diagram

$$\begin{array}{ccccc}
\mathbb{Z}_2 & \to & \mathrm{Sp}(k) \times \mathrm{Sp}(1) & \to & \mathrm{Sp}(k) \cdot \mathrm{Sp}(1) \\
\| \wr & & \downarrow & & \downarrow \\
\mathbb{Z}_2 & \longrightarrow & \mathrm{Spin}(4k) & \longrightarrow & SO(4k)
\end{array}$$

we see that $E \cdot H$ is a spin bundle.

Now the tangent bundle along the fibres T_F of $\mathbb{H}P(E \oplus 1) \to^p Y$ is given by

$$T_F \oplus 1 = (p^*E \oplus 1) \cdot H,$$

and so T_F is clearly spin and p is KR-oriented.

Since $\mathbb{H}P^k = Sp(k+1)/Sp(k) \times Sp(1)$ is a homogeneous space, it has positive scalar curvature relative to the standard metric. By choosing an orthogonal structure on the bundle E, the structure group of $\mathbb{H}P(E \oplus 1) \to^p Y$ reduces to $Sp(k)$, which acts on $\mathbb{H}P^k$ by isometries of the standard metric, hence we can introduce a family of metrics in the fibres all having positive scalar curvature. From Proposition 4.2 and Lichnerowicz's theorem, we see then that $p!(1) = 0$.

(2) The proof in the complex case is similar.

The tangent bundle along the fibres is complex, so p is K-oriented. The symbol class of the Dolbeault complex defines the Thom isomorphism $K(X) \cong K(TX)$, so $p!([H^{-1}])$ is the index of the family of operators

$$\bar{\partial} + \bar{\partial}^*: \Gamma(T^{0,\text{even}} \otimes H^{-1}) \to \Gamma(T^{0,\text{odd}} \otimes H^{-1})$$

in the fibres $\mathbb{C}P^k$. But by Kodaira's vanishing theorem,

$$H^p(\mathbb{C}P^k, \mathcal{O}(H^{-1})) = 0.$$

Since the structure group of the bundle $\mathbb{C}P(E \oplus 1) \to^p Y$ reduces to $U(k)$, we see that the operators in the fibres are all invertible and so $p!([H^{-1}]) = 0$.

Equivalently, we could have used the vanishing theorem for harmonic spinors associated to a Spin^c structure on $\mathbb{C}P^k$ given by Example 2 of **1.2**.

This lemma is essentially an analytic version of the theorems on multiplicativity in fibre bundles of Borel and Hirzebruch [16].

We recall here the *homology* theories associated to K-theory and KR-theory.

Let $X \subset \mathbb{R}^k$ be an embedding with normal bundle N and k large, then homology K-theory is defined by

$$K_m(X) \cong K^{k-m}(N)$$

$$KR_m(X) \cong KR^{k-m}(N).$$

If $f: X \to Y$ is a continuous map of manifolds, then there is a natural transformation

$$f_*: K_m(X) \to K_m(Y).$$

If X^n is weakly almost complex, the Thom class in $K^{k-n}(N)$ defines an *orientation* class $[X] \in K_n(X)$. Similarly, if X is spin, the Bott class defines an orientation class in $KR_n(X)$. The Thom isomorphism theorem then defines the Poincaré duality

$$K_m(X) \cong K^{n-m}(X),$$

and if $f: X^n \to Y^p$ is a K-oriented map, then $f!: K^m(X) \to K^{p-n+m}(Y)$ is defined by f_* via the duality.

Now if $X \to Z^n \to^p Y^m$ is a fibre bundle with X, Y, Z weakly almost complex, it follows from the multiplicative property of the Thom class that

$$p_*[Z] = p!(1)[Y] \in K_n(Y)$$

and similarly for spin manifolds and KR-theory.

Hence we can interpret Lemma 4.8 by saying

$$p_*[\mathbb{H}P(E \oplus 1)] = 0 \in KR_{4k+m}(Y)$$

$$p_*([H^{-1}] \cdot [\mathbb{C}P(E \oplus 1)]) = 0 \in K_{2k+m}(Y)$$

if dim $Y = m$ and Y is spin (resp. weakly almost complex).

To apply the lemma, we now consider blowing up from a differentiable point of view.

Let $Y \subset X$ be a submanifold with (real, complex, or quaternionic) normal bundle N. If we remove a tubular neighborhood N of Y in X and replace it with the Hopf bundle H over $P(N)$ (real, complex, or quaternionic projective bundle) by identification on the boundary $S(N)$, we obtain a new manifold X' by "blowing up along $Y \subset X$," and a "blowing down map" $f: X' \to X$. f restricted to $H \subset X'$ is just the projection $q: N \times_Y P(N) \to N$ restricted to

$$H = \{(x, y) \in N \times_Y P(N) \mid x \in y\} \subset N \times_Y P(N).$$

If X is weakly almost complex and we blow up $Y \subset X$ with complex normal bundle, then X' is weakly almost complex where $H \subset X'$ has the almost complex structure induced from the inclusion $H \subset N \times_Y P(N)$ (this is the almost complex structure which comes from blowing up analytically a complex submanifold $Y \subset X$ where X is a complex manifold).

If X is spin and we blow up $Y \subset X$ with quaternionic normal bundle, then X' is spin.

THEOREM 4.9. (1) *Let $f\colon X' \to X$ be a complex blowing down of weakly almost complex manifolds. Then*

$$f_*[X'] = [X] \in K_n(X).$$

(2) *Let $f\colon X' \to X$ be a quaternionic blowing down of spin manifolds. Then*

$$f_*[X'] = [X] \in KR_n(X).$$

COROLLARY. (1) *The Todd genus of a weakly almost complex manifold is invariant under complex blowing up.*

(2) *The KR-characteristic number $\alpha(X) \in KR^{-n}(\mathrm{pt.})$ of a spin manifold is invariant under quaternionic blowing up. In particular, the $\hat{A}$-genus of a spin $4k$-manifold is invariant.*

Proof of Theorem. (1) We use the homomorphism from unitary bordism to homology K-theory

$$\beta\colon \Omega_p{}^U(X) \to K_p(X)$$

defined as follows. Let $f\colon M^p \to X$ be a mapping of a weakly almost complex manifold M^p to X. Then $\beta([M^p, f]) = f_*[M^p] \in K_p(X)$, where $[M^p] \in K_p(M^p)$ is the orientation class of M.

The Hopf bundle H is diffeomorphic to a tubular neighborhood of $P(N) \subset P(N \oplus 1)$. Consider the map $g\colon P(N \oplus 1) \to D(N)$ defined by

$$g(z, \lambda) = 2z\bar{\lambda}/(\langle z, z\rangle + \lambda\bar{\lambda}),$$

where $(z, \lambda) \in N \oplus 1$. Let

$$A^+ = \{(z, \lambda) \in P(N \oplus 1) \mid \langle z, z\rangle > \lambda\bar{\lambda}\},$$

$$A^- = \{(z, \lambda) \in P(N \oplus 1) \mid \langle z, z\rangle < \lambda\bar{\lambda}\}.$$

Then we have the following commutative diagram:

$$\begin{array}{ccc} A^+ & \xrightarrow{h} & H \subset N \times_Y P(N) \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle \alpha} \\ N & \xrightarrow{\quad\cong\quad} & N, \end{array}$$

where $h(z, \lambda) = (g(z, \lambda), z)$.

h: $A^+ \to H$ and g: $A^- \to N$ are diffeomorphisms, hence g represents the blowing down map if we identify A^+ and H by h. Note that h is antilinear in the normal direction to $P(N) \subset P(N \oplus 1)$ and so the complex structure induced on A^+ by the diffeomorphism h is obtained by taking the conjugate of the complex structure in the normal bundle of $P(N) \subset P(N \oplus 1)$. The tangent bundle along the fibres of $P(N \oplus 1)$ has the standard stable complex structure given by

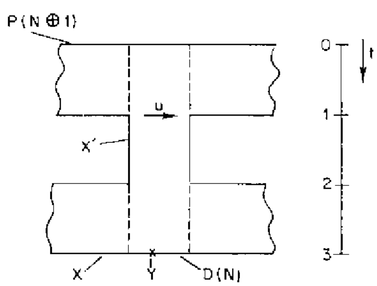
$$T_F \oplus 1 \cong H \otimes (p^*N \oplus 1),$$

where p: $P(N \oplus 1) \to Y$ is the projection. The complex structure given by

$$T_F \oplus 1 \cong (H \otimes p^*N) \oplus \bar{H} \tag{1}$$

induces the required weakly almost comlpex structure on A^+.

Let Z be the following manifold with boundary, after straightening the angles:



We have a map j: $Z \to X$ given by

$$\begin{aligned} j(x, t) &= g(x) && 0 \leqslant t \leqslant 1 \\ &= 2u/(t + (2 - t)\langle u, u\rangle) && 1 \leqslant t \leqslant 2 \\ &= x && 2 \leqslant t \leqslant 3 \end{aligned}$$

where $u = z\lambda^{-1} \in D(N)$.

We extend the weakly almost complex structure on the boundary to the interior and then we obtain a relation in $\Omega_n{}^U(X)$, namely,

$$[X', f] - [X, \mathrm{id}] = [\tilde{P}(N \oplus 1), g],$$

where $\tilde{P}(N \oplus 1)$ denotes $P(N \oplus 1)$ with the weakly almost complex structure given by $T \oplus 1 \simeq p^*T_Y \oplus (H \oplus p^*N) \oplus \bar{H}$. Using the homomorphism β, we see that

$$f_*[X'] - [X] = g_*[\tilde{P}(N \oplus 1)] \in K_n(X).$$

But $g\colon P(N \oplus 1) \to D(N) \hookrightarrow X$ retracts to $\tilde{p}\colon P(N \oplus 1) \to^p Y \hookrightarrow X$, so to prove the theorem we have to show $p_*[\tilde{P}(N \oplus 1)] = 0 \in K_n(Y)$. Now

$$\lambda^p(E \oplus H) = \lambda^p E \oplus (\lambda^{p-1}E \otimes H),$$

$$\lambda^p(E \oplus H^{-1}) = \lambda^p E \oplus (\lambda^{p-1}E \otimes H^{-1}).$$

Hence,

$$\lambda^{\mathrm{odd}}(E \oplus H^{-1}) \simeq H^{-1} \otimes \lambda^{\mathrm{even}}(E \oplus H),$$

$$\lambda^{\mathrm{even}}(E \oplus H^{-1}) \simeq H^{-1} \otimes \lambda^{\mathrm{odd}}(E \oplus H),$$

and so the Todd class of the stable complex structure on T_F given by (i) is the standard one multiplied by $-[H^{-1}]$.

Hence, the orientation class $[\tilde{P}(N \oplus 1)] = -[H]^{-1}[P(N \oplus 1)]$ and by Lemma 4.8 $p_*[\tilde{P}(N \oplus 1)] = 0$.

(2) The proof of part 2 is similar. We use the homomorphism:

$$\tilde{\beta}\colon \Omega_p^{\mathrm{Spin}}(X) \to KR_p(X)$$

and part 1 of Lemma 4.8.

Note that $g\colon P(N \oplus 1) \to D(N)$ is well-defined for quaternionic projective space. $\mathbb{H}P^n$ is defined as the equivalence classes of $\mathbb{H}^{n+1} - \{0\}$ under *right* multiplication by a quaternion w. Then,

$$(zw)(\overline{\lambda w}) = zw\bar{w}\bar{\lambda} = z(w\bar{w})\bar{\lambda} = (w\bar{w})\, z\bar{\lambda},$$

and so g is well-defined.

Proof of Corollary. The invariants are given by mapping to a point $h\colon X \to \text{pt.}$

$$T(X^{2n}) = h_*[X^{2n}] \in K_{2n}(\text{pt.}) \cong \mathbb{Z},$$
$$\alpha(X) = h_*[X] \in KR_n(\text{pt.}) \cong KR^{-n}(\text{pt.}).$$

Now if $f\colon X' \to X$ is the blowing down map, $h'\colon X' \to \text{pt.}$ is given by $h' = hf$ and hence

$h_*'[X'] = h_* f_*[X'] = h_*[X]$ by the theorem which proves the corollary.

Note Added in Proof. Theorem 1.2 is incorrect as it stands as we only calculated the principal isotropy subgroup. In fact, Berger's classification shows that an irreducible factor of the holonomy group must lie in $SU(n)$, G_2, or Spin(7) so modulo factors in dimension 7 or 8 the theorem holds.

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